# Bootstrap percolation on the Hamming graphs

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**Abstract**

The $r$-edge bootstrap percolation on a graph is an activation process of the edges. The process starts with some initially activated edges and then, in each round, any inactive edge whose one of endpoints is incident to at least $r$ active edges becomes activated. A set of initially activated edges leading to the activation of all edges is said to be a percolating set. Denote the minimum size of a percolating set in the $r$-edge bootstrap percolation process on a graph $G$ by $m_e(G, r)$. The importance of the $r$-edge bootstrap percolation relies on the fact that $m_e(G, r)$ provides bounds on $m(G, r)$, that is, the minimum size of a percolating set in the $r$-neighbor bootstrap percolation process on $G$. In this paper, we explicitly determine $m_e(K_n^d, r)$, where $K_n^d$ is the Cartesian product of $d$ copies of the complete graph on $n$ vertices which is referred as Hamming graph. Using this, we show that $m(K_n^d, r) = (1 + o(1))\frac{d^{r-1}}{r!}$ when $n, r$ are fixed and $d$ goes to infinity which extends a known result on hypercubes.

**Keywords:** Bootstrap percolation, Hamming graph, Percolating set.

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## 1. Introduction

Bootstrap percolation processes on graphs can be interpreted as a family of cellular automata, a concept introduced in 1966 by von Neumann [14]. They have been extensively investigated in several diverse fields such as combinatorics, probability theory, statistical physics and social sciences. The $r$-neighbor bootstrap percolation is the most studied of such processes which was firstly introduced in 1979 by Chalupa, Leath and Reich [8]. This process has also been treated in

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the literature under other names like irreversible threshold, influence propagation and dynamic monopoly.

Throughout this paper, all graphs are assumed to be finite, undirected, without loops and multiple edges. For a graph $G$, the vertex set of $G$ is denoted by $V(G)$ and the edge set of $G$ is denoted by $E(G)$. The $r$-neighbor bootstrap percolation can be defined formally as follows. Given a nonnegative integer $r$ and a graph $G$, the *$r$-neighbor bootstrap percolation process* on $G$ begins with a subset $V_0$ of initially activated vertices of $G$ and then, at step $i$ of the process, the set $V_i$ of active vertices is

$$V_i = V_{i-1} \cup \left\{ v \in V(G) \;\middle|\; \begin{array}{l} \text{The vertex } v \text{ is adjacent to} \\ \text{at least } r \text{ vertices in } V_{i-1}. \end{array} \right\}$$

for each $i \geqslant 1$. We say $V_0$ is a *percolating set* of $G$ if $V_t = V(G)$ for some $t \geqslant 0$. An extremal problem here is to determine the minimum size of a percolating set which is denoted by $m(G, r)$. The size of percolating sets has been studied for various families of graphs such as hypercubes [13], grids [4], trees [15] and random graphs [9].

An edge version of the $r$-neighbor bootstrap percolation can be defined by considering a special case of the so-called 'graph bootstrap percolation'. The concept of graph bootstrap percolation was firstly introduced in 1968 by Bollobás under a different name [7] and was later studied in 2012 by Balogh, Bollobás and Morris under the current name [3]. Graph bootstrap percolation can be defined formally as follows. Given two graphs $G$ and $H$, the *$H$-bootstrap percolation process* on $G$ begins with a subset $E_0$ of initially activated edges of $G$ and then, at step $i$ of the process, the set $E_i$ of active edges is

$$E_i = E_{i-1} \cup \left\{ e \in E(G) \;\middle|\; \begin{array}{l} \text{There exists a subgraph } H_e \text{ of } G \text{ such} \\ \text{that } H_e \text{ is isomorphic to } H,\ e \in E(H_e) \\ \text{and } E(H_e) \setminus \{e\} \subseteq E_{i-1}. \end{array} \right\}$$

for each $i \geqslant 1$. The set $E_0$ is called a *percolating set* of $G$ if $E_t = E(G)$ for some $t \geqslant 0$. The minimum size of a percolating set in the $H$-bootstrap percolation process on $G$ is equal to the so-called *weak saturation number* of $H$ in $G$ and is denoted by $\mathrm{wsat}(G, H)$. Denoting the star graph on $m$ edges by $S_m$, we refer to the $S_{r+1}$-bootstrap percolation as the *$r$-edge bootstrap percolation* which can be considered as an edge analogue of the $r$-neighbor bootstrap percolation. For simplicity, we write $m_e(G, r)$ instead of $\mathrm{wsat}(G, S_{r+1})$. The 2-edge bootstrap percolation had been studied in 1984 by Lenormand and Zarcone under a different name [12]. By a result from [11], we have

$$\frac{m_e(G,r)}{r} \leqslant m(G,r) \leqslant m_e(G,r) + \Big| \big\{ v \in V(G) \;\big|\; \deg(v) < r \big\} \Big|, \tag{1}$$

where $\deg(v) = |\{x \in V(G) \,|\, x \text{ is adjacent to } v\}|$.

Let us fix here some notation and terminology used in the rest of the paper. For every two adjacent vertices $u$ and $v$, the edge joining $u$ and $v$ is denoted by $uv$ or $\{u, v\}$. The *Cartesian product* of two graphs $G$ and $H$, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ in which two vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent if and only if either $g_1 = g_2$ and $h_1 h_2 \in E(H)$ or $g_1 g_2 \in E(G)$ and $h_1 = h_2$. Denote by $G^d$ the Cartesian product of $d$ vertex disjoint copies of the graph $G$. For any integer $n$, we let $[\![n]\!] = \{0, 1, \ldots, n-1\}$ if $n \geqslant 1$ and $[\![n]\!] = \varnothing$ otherwise.

We denote the complete graph on $n$ vertices by $K_n$ and we always consider $[\![n]\!] = \{0, 1, \ldots, n-1\}$ as the vertex set of $K_n$. The graph $K_n^d$ is called a *Hamming graph* of dimension $d$.

Balister, Bollobás, Lee and Narayanan [1] gave the lower bound $(r/d)^d$ and the upper bound $r^d/(2d!)$ on $m(K_n^d, r)$ for $n \geqslant 2$. This along with (1) motivates to study $m_e(K_n^d, r)$. In the current paper, we apply the polynomial technique introduced by Hambardzumyan, Hatami and Qian [11] to get an explicit formula for $m_e(K_n^d, r)$ for all values of $n, r, d$. Note that $m_e(G, r) = |E(G)|$ if $r \geqslant \max\{\deg(v) \,|\, v \in V(G)\}$. In particular, $m_e(K_n^d, r) = \frac{(n-1)d}{2} n^d$ if $r \geqslant (n-1)d$.

**Theorem 1.1.** *Let $n \geqslant 2$ and $d, r \geqslant 0$ be three integers with $0 \leqslant r \leqslant (n-1)d$. Then,*

$$m_e\left(K_n^d, r\right) = \sum_{i_1=0}^{r-1} \sum_{i_2=0}^{r-i_1-1} \cdots \sum_{i_{n-1}=0}^{r-i_1-\cdots-i_{n-2}-1} (r - i_1 - \cdots - i_{n-1}) \binom{d}{i_1} \binom{i_1}{i_2} \cdots \binom{i_{n-2}}{i_{n-1}}.$$

Bidgoli, Mohammadian and Tayfeh-Rezaie [6] had proved that

$$m_e(K_n^d, r) = \binom{d+r}{d+1}$$

if $0 \leqslant r \leqslant n-1$. In the current paper, we particularly show that

$$m_e\left(K_n^d, r\right) = \binom{nd-r}{d+1} + \left(r - \frac{(n-1)d}{2}\right) n^d$$

if $(n-1)(d-1) \leqslant r \leqslant (n-1)d$.

An explicit formula for $m_e(K_2^d, r)$ was already found by Morrison and Noel [13] which gives the asymptotic formula $m(K_2^d, r) = \frac{1+o(1)}{r} \binom{d}{r-1}$ when $r$ is fixed and $d$ goes to infinity, settling a conjecture raised by Balogh and Bollobás [2]. We generalize the asymptotic result by establishing the following theorem as a consequence of Theorem 1.1.

**Theorem 1.2.** *Let $n \geqslant 2$ and $r$ be two fixed positive integers and let $d$ be an integer tending to infinity. Then,*

$$m\left(K_n^d, r\right) = \big(1 + o(1)\big) \frac{d^{r-1}}{r!}.$$

Another asymptotic result, which was proved by Bidgoli, Mohammadian and Tayfeh-Rezaie [6], states that $m(K_n^d, r) = \frac{1+o(1)}{(d+1)!} r^d$ when both $r, d$ go to infinity with $d = o(\sqrt{r})$ and $n \geqslant r + 1$. Furthermore, recursive formulas for $m_e(P_n^d, r)$ and $m_e(C_n^d, r)$ are found in [11], where $P_n$ is the path graph on $n$ vertices and $C_n$ is the cycle graph on $n$ vertices.

The rest of the paper is organized as follows. In Section 2, we recall the polynomial technique which is used to get a lower bound on $m_e(G, r)$ and we present a new proof for it. In Section 3, we present an explicit formula for $m_e(K_n^d, r)$ for all values of $n, r, d$. Using this, we present an asymptotic formula for $m(K_n^d, r)$ when $n, r$ are fixed and $d$ tends to infinity.

## 2. The algebraic method

In this section, we recall a polynomial technique which is introduced by Hambardzumyan, Hatami and Qian [11] and we will use it to get a lower bound on $m_e(K_n^d, r)$ in the next section.

We show here that the polynomial technique can be regarded as a special case of a general framework due to Balogh, Bollobás, Morris and Riordan [5]. A short proof of the following interesting lemma is given in [10].

**Theorem 2.1** ([5]). *Let $G, F$ be two graphs and let $W$ be an arbitrary vector space. Assume that there is a subset $\{w_e \,|\, e \in E(G)\}$ of $W$ such that for each copy $F'$ of $F$ in $G$ there are nonzero scalars $\{\lambda_{e,F'} \,|\, e \in E(F')\}$ such that $\sum_{e\in E(F')} \lambda_{e,F'} w_e = 0$. Then,*

$$\mathrm{wsat}(G, F) \geqslant \dim \Big(\mathrm{span}\big\{w_e \,\big|\, e \in E(G)\big\}\Big).$$

We note that the following definition is slightly different from the original version.

**Definition 2.2** ([11]). Let $r$ be a nonnegative integer and let $G$ be a graph equipped with a proper edge coloring $c : E(G) \to \mathbb{R}$. Let $W_c(G, r)$ be the vector space over $\mathbb{R}$ consisting of all functions $\varphi : E(G) \to \mathbb{R}$ for which there exist polynomials $\{P_v(x)\}_{v\in V(G)}$ satisfying

(i) $\deg P_v(x) \leqslant r - 1$ for any vertex $v \in V(G)$;

(ii) $P_u(c(uv)) = P_v(c(uv)) = \varphi(uv)$ for each edge $uv \in E(G)$.

It is said that the polynomials $\big\{P_v(x)\big\}_{v\in V(G)}$ *recognize* $\varphi$. Notice that we adopt the convention that the degree of the zero polynomial is $-1$.

The next theorem provides an interesting linear algebraic lower bound on $m_e(G, r)$. Some other nice applications of vector spaces and polynomials for bootstrap percolation processes on graphs can be found in [1, 5, 6, 10, 11, 13]. Indeed, the literature is full of proofs via linear and multilinear algebraic techniques for which no combinatorial proof is known.

**Theorem 2.3** ([11]). *Let $r$ be a nonnegative integer and let $c : E(G) \to \mathbb{R}$ be a proper edge coloring of a graph $G$. Then, $m_e(G, r) \geqslant \dim W_c(G, r)$.*

**Proposition 2.4.** *Theorem 2.3 can be proved by Theorem 2.1.*

*Proof.* Let $\{\varphi_1, \ldots, \varphi_k\}$ be a basis for $W_c(G, r)$. We assign to every edge $e \in E(G)$ a vector $w_e = (\varphi_1(e), \ldots, \varphi_k(e))$ in $\mathbb{R}^k$. We claim that for every copy of $S_{r+1}$ in $G$ with the central vertex $v$ and the edge set $\{e_1, \ldots, e_{r+1}\}$ there are nonzero scalars $\lambda_1, \ldots, \lambda_{r+1}$ such that $\sum_{i=1}^{r+1} \lambda_i w_{e_i} = 0$. We know that there are nonzero scalers $\lambda_1, \ldots, \lambda_{r+1}$ such that

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_{r+1} \\ c_1^2 & c_2^2 & \cdots & c_{r+1}^2 \\ \vdots & \vdots & \vdots & \vdots \\ c_1^{r-1} & c_2^{r-1} & \cdots & c_{r+1}^{r-1} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{r+1} \end{bmatrix} = 0,$$

where $c_i = c(e_i)$. Therefore, for any polynomial $P(x)$ of degree $r - 1$, we have

$$\begin{bmatrix} P(c_1) & P(c_2) & \cdots & P(c_{r+1}) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{r+1} \end{bmatrix} = 0$$

and so

$$\begin{bmatrix} P_v^{(1)}(c_1) & P_v^{(1)}(c_2) & \cdots & P_v^{(1)}(c_{r+1}) \\ \vdots & \vdots & \vdots & \vdots \\ P_v^{(r+1)}(c_1) & P_v^{(r+1)}(c_2) & \cdots & P_v^{(r+1)}(c_{r+1}) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{r+1} \end{bmatrix} = 0, \tag{2}$$

where $P_v^{(i)}$ is the polynomial corresponding to the vertex $v$ and the function $\varphi_i$ for $i = 1, \ldots, r+1$. Equality (2) is equivalent to $\sum_{i=1}^{r+1} \lambda_i w_{e_i} = 0$, as claimed. Now, using Theorem 2.1, we have

$$m_e(G, r) = \mathrm{wsat}(G, S_{r+1}) \geqslant \dim \Big( \mathrm{span}\big\{ w_e \,\big|\, e \in E(G) \big\} \Big) = k. \qquad \square$$

## 3. Percolating sets of Hamming graphs

In this section, we first present an explicit formula for $m_e(K_n^d, r)$ for all positive integers $n, r, d$ and then, as a consequence, we give an asymptotic formula for $m(K_n^d, r)$ when $n, r$ are fixed and $d$ tends to infinity. Throughout this section, for every integers $n, r, d$, we let $f = r-1-(n-1)(d-1)$ and

$$g = \begin{cases} 1 & \text{if } f \leqslant -2, \\ f+2 & \text{if } -1 \leqslant f \leqslant n-2, \\ n & \text{if } f \geqslant n-1. \end{cases}$$

Throughout this section, $m_e(G, i)$ and $W_c(G, i)$ are respectively interpreted as 0 and $\{0\}$ for any graph $G$ if $i < 0$.

**Lemma 3.1.** *Let $n, r, d$ be three positive integers. Then,*

$$m_e\left(K_n^d, r\right) \leqslant \sum_{i=0}^{n-1} m_e\left(K_n^{d-1}, r-i\right) + \binom{g}{2} n^{d-1}. \tag{3}$$

*Proof.* If $r \geqslant (n-1)d$, then $g = n$ and

$$m_e\left(K_n^d, r\right) = \left|E\left(K_n^d\right)\right| = \sum_{i=0}^{n-1} \left|E\left(K_n^{d-1}\right)\right| + \binom{n}{2} n^{d-1} = \sum_{i=0}^{n-1} m_e\left(K_n^{d-1}, r-i\right) + \binom{g}{2} n^{d-1}.$$

So, assume that $r < (n-1)d$. For any $i \in [\![n]\!]$, consider the subgraph $H_i$ of $K_n^d$ induced on $\{(v,i) \in V(K_n^d) \,|\, v \in V(K_n^{d-1})\}$ which is clearly isomorphic to $K_n^{d-1}$. Also, consider a percolating set $E_i$ of the minimum possible size in the $(r-i)$-edge bootstrap percolation process on $H_i$ and activate its elements. Let

$$E = \left\{ \{(v,i),(v,j)\} \in E\left(K_n^d\right) \,\middle|\, v \in V\left(K_n^{d-1}\right) \text{ and } 0 \leqslant i < j \leqslant g-1 \right\}$$

and active all edges in $E$. Thus, the number of all initially activated edges is

$$\sum_{i=0}^{n-1} |E_i| + |E| = \sum_{i=0}^{n-1} m_e\left(K_n^{d-1}, r-i\right) + \binom{g}{2} n^{d-1}.$$

Now, for any $i \in [\![n]\!]$, we show that all the edges incident to the vertices of $H_i$ become activated in the $r$-edge bootstrap percolation process consecutively. Assume that all the edges incident to the vertices in $\bigcup_{j=0}^{i-1} V(H_j)$ became activated in the $r$-edge bootstrap percolation process. Consider a vertex $x = (v, i) \in V(H_i)$. The vertex $x$ is connected to $i$ vertices $(v, 0), \ldots, (v, i-1)$ by activated edges in $E$ and also $x$ is adjacent to $r - i$ vertices in $V(H_i)$ by activated edges in $E_i$. Therefore, $x$ is incident to $r$ activated edges and so all other edges incident to $x$ in $E(H_i)$ can be activated in the $r$-edge bootstrap percolation process. Since $x$ is incident to $g - i - 1$ activated edges in $E$, the number of activated edges incident to $x$ is $i + (n-1)(d-1) + g - i - 1$ in $E(K_n^d)$. Since $g = \max\{1, f+2\}$ and $(n-1)(d-1) + g - 1 = \max\big\{(n-1)(d-1), r\big\} \geqslant r$, all edges incident to $x$ in $K_n^d$ can be activated in the $r$-edge bootstrap percolation process. This shows that $\bigcup_{i=0}^{n-1} E_i \cup E$ is a percolating set of desired size in the $r$-edge bootstrap percolation process on $K_n^d$. □

**Lemma 3.2.** *Let $n, r, d$ be three positive integers and let $c : E(K_n^{d-1}) \to \mathbb{R}$ be a proper edge coloring of $K_n^{d-1}$. Then, there is a proper edge coloring $\widehat{c} : E(K_n^d) \to \mathbb{R}$ such that*

$$\dim W_{\widehat{c}}\left(K_n^d, r\right) \geqslant \sum_{i=0}^{n-1} \dim W_c\left(K_n^{d-1}, r-i\right) + \binom{g}{2} n^{d-1}. \tag{4}$$

*Proof.* Consider arbitrary mutually distinct nonzero real numbers $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ such that none of the numbers $\gamma_i\gamma_j$ is in the image of $c$. For every two adjacent vertices $u = (a, i)$ and $v = (b, j)$ of $K_n^{d-1} \square K_n$, define

$$\widehat{c}(uv) = \begin{cases} c(ab) & \text{if } i = j, \\ \gamma_i\gamma_j & \text{if } a = b. \end{cases}$$

It is straightforward to check that $\widehat{c} : E(K_n^d) \to \mathbb{R}$ is a proper edge coloring. Fix $k \in [\![n]\!]$, a basis $\mathscr{B}_k$ for $W_c(K_n^{d-1}, r-k)$, and a function $\varphi \in \mathscr{B}_k$. According to Definition 2.2, there exist polynomials $\{P_a^{\varphi}(x)\}_{a \in V(K_n^{d-1})}$ recognizing $\varphi$. Define polynomial $Q_u^{k,\varphi}$ for any vertex $u = (a, i) \in V(K_n^{d-1} \square K_n)$ as $Q_u^{k,\varphi}(x) = P_a^{\varphi}(x) T_i^k(x)$, where

$$T_i^k(x) = \prod_{\ell=0}^{k-1} (\gamma_i - \gamma_\ell)\left(\frac{x}{\gamma_i} - \gamma_\ell\right).$$

Since the degree of $P_a^{\varphi}(x)$ is at most $r - k - 1$ and the degree of $T_i^k(x)$ is $k$, we conclude that the degree of $Q_u^{k,\varphi}$ is at most $r - 1$. Note that $T_i^k(\gamma_i\gamma_j) = T_j^k(\gamma_i\gamma_j)$ for all $i$ and $j$. Also, we know from Definition 2.2 that $P_a^{\varphi}(c(ab)) = P_b^{\varphi}(c(ab))$ for each edge $ab \in E(K_n^{d-1})$. Hence, $Q_u^{k,\varphi}$ and $Q_v^{k,\varphi}$ have the same value on $\widehat{c}(uv)$ for any edge $uv \in E(K_n^d)$. This implies that $\{Q_u^{k,\varphi}\}_{u \in V(K_n^d)}$ recognize a function $\Phi_{k,\varphi} \in W_{\widehat{c}}(K_n^d, r)$. Now, assume that $f \geqslant 0$. Define polynomial $R_u^{s,t,y}$ for every integers $s, t \in [\![g]\!]$ and vertices $y \in V(K_n^{d-1})$, $u = (a, i) \in V(K_n^{d-1} \square K_n)$ as $R_u^{s,t,y}(x) = S_a^y(x) L_i^{s,t}(x)$, where

$$S_a^y(x) = \begin{cases} 0 & \text{if } y \neq a, \\ \displaystyle\prod_{yz \in E\left(K_n^{d-1}\right)} (x - c(yz)) & \text{if } y = a. \end{cases}$$

and

$$L_i^{s,t}(x) = \begin{cases} 0 & \text{if } i \in [\![g]\!] \setminus \{s,t\}, \\ \displaystyle\prod_{\ell \in [\![g]\!] \setminus \{s,t\}} \frac{x - \gamma_i \gamma_\ell}{\gamma_s \gamma_t - \gamma_i \gamma_\ell} & \text{if } i \in \{s,t\}, \\ \displaystyle\prod_{\ell \in [\![g]\!] \setminus \{s,t\}} \frac{(\gamma_i - \gamma_\ell)\left(\dfrac{x}{\gamma_i} - \gamma_\ell\right)}{(\gamma_s - \gamma_\ell)(\gamma_t - \gamma_\ell)} & \text{if } i \in [\![n]\!] \setminus [\![g]\!]. \end{cases}$$

As $f \geqslant 0$, we find that $r - 1 \geqslant (n-1)(d-1)$ and, since $g = \min\{n, \max\{1, f+2\}\}$, the degree of every polynomial $R_u^{s,t,y}$ is

$$(n-1)(d-1) + g - 2 = \min\Big\{(n-1)d - 1, \max\big\{(n-1)(d-1) - 1, r-1\big\}\Big\} \leqslant r - 1.$$

Let $u = (a,i)$ and $v = (b,j)$ be two arbitrary distinct vertices in $V(K_n^{d-1} \square K_n)$. Note that $S_a^y(c(ab)) = S_b^y(c(ab)) = 0$ for all $ab \in E(K_n^{d-1})$. This means that $R_u^{s,t,y}(\widehat{c}(uv)) = R_v^{s,t,y}(\widehat{c}(uv))$ if $i = j$. Also, it is easy to check that $L_i^{s,t}(\gamma_i\gamma_j) = L_j^{s,t}(\gamma_i\gamma_j)$ for all distinct $i, j \in [\![n]\!]$. This means that $R_u^{s,t,y}(\widehat{c}(uv)) = R_v^{s,t,y}(\widehat{c}(uv))$ if $a = b$. Therefore, $\{R_u^{s,t,y}\}_{u \in V(K_n^d)}$ recognize a function $\Psi_{s,t,y} \in W_{\widehat{c}}(K_n^d, r)$.

Since the number of functions $\Phi_{k,\varphi}$ is $\sum_{i=0}^{n-1} \dim W_c(G, r-i)$ and the number of functions $\Psi_{s,t,y}$ is $\binom{g}{2} n^{d-1}$, it remains to show that all functions $\Phi_{k,\varphi}$ and $\Psi_{s,t,y}$ are linearly independent. Suppose that

$$\sum_{k,\varphi} \lambda_{k,\varphi} \Phi_{k,\varphi} + \sum_{s,t,y} \mu_{s,t,y} \Psi_{s,t,y} = 0 \tag{5}$$

for some scalars $\lambda_{k,\varphi}, \mu_{s,t,y} \in \mathbb{R}$. Let $u = (a,i)$ and $v = (b,j)$ be two adjacent vertices of $K_n^{d-1} \square K_n$. If $i = j$, then $\widehat{c}(uv) = c(ab)$ and since $S_a^y(c(ab)) = 0$ we find from (5) that

$$\sum_{k,\varphi} \lambda_{k,\varphi} \Phi_{k,\varphi}(uv) = \sum_{s,t,y} \mu_{s,t,y} \Psi_{s,t,y}(uv) = 0. \tag{6}$$

If $a = b$, then $\widehat{c}(uv) = \gamma_i \gamma_j$ and since $T_i^k(\gamma_i\gamma_j) = 0$ we find from (5) that

$$\sum_{k,\varphi} \lambda_{k,\varphi} \Phi_{k,\varphi}(uv) = \sum_{s,t,y} \mu_{s,t,y} \Psi_{s,t,y}(uv) = 0. \tag{7}$$

From (6) and (7) we find that

$$\sum_{k,\varphi} \lambda_{k,\varphi} \Phi_{k,\varphi} = \sum_{s,t,y} \mu_{s,t,y} \Psi_{s,t,y} = 0.$$

First, we show that all scalars $\lambda_{k,\varphi}$ are equal to zero. Towards a contradiction, suppose that $k_0$ is the smallest number from $[\![n]\!]$ such that $\lambda_{k_0,\varphi} \neq 0$ for some $\varphi$. Note that, for every $i < k$, the term $\gamma_i - \gamma_i$ appears in the expression of $T_i^k$ and so $T_i^k = 0$. This yields that $Q_u^{k,\varphi} = 0$ for any vertex $u=(a,k_0)\in V(K_n^{d-1}\square K_n)$ and integer $k>k_0$. Thus, for any two adjacent vertices $u=(a,k_0)$ and $v=(b,k_0)$ of $K_n^{d-1}\square K_n$, we have

$$0=\sum_{k,\varphi}\lambda_{k,\varphi}\Phi_{k,\varphi}(uv)=\sum_{k,\varphi}\lambda_{k,\varphi}Q_u^{k,\varphi}\big(\widehat{c}(uv)\big)=\sum_{\varphi\in\mathscr{B}_{k_0}}\lambda_{k_0,\varphi}P_a^{\varphi}\big(c(ab)\big)T_{k_0}^{k_0}\big(c(ab)\big).$$

Our assumption on $\gamma_0,\gamma_1,\dots,\gamma_{n-1}$ implies that $T_{k_0}^{k_0}(c(ab))\neq 0$. Therefore,

$$\left(\sum_{\varphi\in\mathscr{B}_{k_0}}\lambda_{k_0,\varphi}\varphi\right)(ab)=\sum_{\varphi\in\mathscr{B}_{k_0}}\lambda_{k_0,\varphi}P_a^{\varphi}\big(c(ab)\big)=0$$

for each edge $ab\in E(K_n^{d-1})$. This means that

$$\sum_{\varphi\in\mathscr{B}_{k_0}}\lambda_{k_0,\varphi}\varphi=0$$

which is a contradiction, since $\mathscr{B}_{k_0}$ is a basis for $W_c(G,r-k_0)$.

Next, we show that all scalars $\mu_{s,t,y}$ are equal to zero. We have $L_i^{s,t}(\gamma_i\gamma_j)=1$ if $(i,j)=(s,t)$ and $L_i^{s,t}(\gamma_i\gamma_j)=0$ if $(i,j)\neq(s,t)$. To verify this, note that if $i=s$, then $j\neq t$ and so the term $(\gamma_j-\gamma_j)/(\gamma_t-\gamma_j)$ appears in the expression of $L_i^{s,t}(\gamma_i\gamma_j)$, implying that $L_i^{s,t}(\gamma_i\gamma_j)=0$. Also, for every $i,j\in[\![g]\!]$, we have $S_a^y(\gamma_i\gamma_j)\neq 0$ if and only if $y=a$. Therefore, for every integers $s_0,t_0\in[\![g]\!]$ and vertex $y_0\in V(K_n^{d-1})$, by letting $u=(y_0,s_0)$ and $v=(y_0,t_0)$, we have

$$0=\left(\sum_{s,t,y}\mu_{s,t,y}\Psi_{s,t,y}\right)(uv)=\mu_{s_0,t_0,y_0}S_{y_0}^{y_0}(\gamma_{s_0}\gamma_{t_0}).$$

Since $S_{y_0}^{y_0}(\gamma_{s_0}\gamma_{t_0})\neq 0$, we conclude that $\mu_{s_0,t_0,y_0}=0$. Hence, we proved that all scalars $\lambda_{k,\varphi}$ and $\mu_{s,t,y}$ are zero. This completes the proof. □

The following theorem particularity demonstrates that the equalities hold in (3) and (4).

**Theorem 3.3.** *Let $n,d$ be two positive integers. Then, there is a proper edge coloring $c_{n,d}:E(K_n^d)\to\mathbb{R}$ such that $m_e(K_n^d,r)=\dim W_{c_{n,d}}(K_n^d,r)$ for any nonnegative integer $r$. Moreover,*

$$m_e\left(K_n^d,r\right)=\sum_{i=0}^{n-1}m_e\left(K_n^{d-1},r-i\right)+\binom{g}{2}n^{d-1}. \tag{8}$$

*Proof.* We prove by induction on $d$ that $m_e(K_n^d,r)=\dim W_{c_{n,d}}(K_n^d,r)$ for a proper edge coloring $c_{n,d}:E(K_n^d)\to\mathbb{R}$. Note that the latter equality trivially holds if $d$ is replaced by 0. So, assume that there exists a proper edge coloring $c_{n,d-1}:E(K_n^{d-1})\to\mathbb{R}$ such that $m_e(K_n^{d-1},r)=\dim W_{c_{n,d-1}}(K_n^{d-1},r)$ for all positive integers $n,r$. By Theorem 2.3, Lemma 3.1 and Lemma 3.2, there is a proper edge coloring $c_{n,d}:E(K_n^d)\to\mathbb{R}$ such that

$$m_e\left(K_n^d,r\right)\geqslant\dim W_{c_{n,d}}\left(K_n^d,r\right)$$

$$\geqslant \sum_{i=0}^{n-1} \dim W_{c_{n,d-1}}\left(K_n^{d-1}, r-i\right) + \binom{g}{2} n^{d-1}$$
$$= \sum_{i=0}^{n-1} m_e\left(K_n^{d-1}, r-i\right) + \binom{g}{2} n^{d-1}$$
$$\geqslant m_e\left(K_n^d, r\right),$$

meaning that $m_e(K_n^d, r) = \dim W_{c_{n,d}}(K_n^d, r)$. The 'moreover' statement is straightforwardly valid from (3) and (4). □

Now, we are going to prove Theorem 1.1. For this, we need some lemmas. At the beginning, we establish the following lemma as mentioned in the introduction.

**Lemma 3.4.** *Let $n \geqslant 2$ and $d, r \geqslant 0$ be three integers with $(n-1)(d-1) \leqslant r \leqslant (n-1)d$. Then,*

$$m_e\left(K_n^d, r\right) = \binom{nd-r}{d+1} + \left(r - \frac{(n-1)d}{2}\right) n^d. \tag{9}$$

*Proof.* We proceed with the proof by induction on $d$. If $d = 0$, then $r = 0$ and (9) is obviously valid. Let $d \geqslant 1$ and assume that (9) holds for $d-1$. Since $(n-1)(d-1) \leqslant r \leqslant (n-1)d$ and $f = r - 1 - (n-1)(d-1)$, we find that $-1 \leqslant f \leqslant n-2$ and so $g = f + 2$. We derive from (8) that

$$\begin{aligned}
m_e\left(K_n^d, r\right) &= \sum_{i=0}^{n-1} m_e\left(K_n^{d-1}, r-i\right) + \binom{f+2}{2} n^{d-1} \\
&= \sum_{i=0}^{f+1} m_e\left(K_n^{d-1}, r-i\right) + \sum_{i=f+2}^{n-1} m_e\left(K_n^{d-1}, r-i\right) + \binom{f+2}{2} n^{d-1} \\
&= \sum_{i=0}^{f+1} \frac{(n-1)(d-1)n^{d-1}}{2} \\
&+ \sum_{i=f+2}^{n-1} \left(\binom{n(d-1)-(r-i)}{d} + \left((r-i) - \frac{(n-1)(d-1)}{2}\right) n^{d-1}\right) \\
&+ \binom{f+2}{2} n^{d-1} \\
&= \sum_{i=d}^{nd-r-1} \binom{i}{d} + \left(r - \frac{(n-1)d}{2}\right) n^d \\
&= \binom{nd-r}{d+1} + \left(r - \frac{(n-1)d}{2}\right) n^d,
\end{aligned}$$

where the last equality is obtained from the well known combinatorial identity $\sum_{i=0}^{k} \binom{m+i}{m} = \binom{m+k+1}{m+1}$. This establishes (9) and completes the proof. □

We need the following combinatorial identity in the proof of next lemma.

**Proposition 3.5.** *For every two integers $m \geqslant 0$ and $k \geqslant 1$,*

$$\sum_{i_1=0}^{k} \cdots \sum_{i_{k-1}=0}^{k} \sum_{i_k=0}^{k-i_1-\cdots-i_{k-1}} \binom{m}{i_k}\binom{i_k}{i_{k-1}} \cdots \binom{i_2}{i_1} = \binom{m+k}{m}.$$

*Proof.* It is well known that $\sum_{i=0}^{k} \binom{m}{i}\binom{k}{k-i} = \binom{m+k}{k}$. By using this equality repeatedly, we have

$$\begin{aligned}
\binom{m+k}{k} &= \sum_{i_1=0}^{k} \binom{m}{i_1}\binom{k}{k-i_1} \\
&= \sum_{i_1=0}^{k} \binom{m}{i_1} \sum_{i_2=0}^{k-i_1} \binom{i_1}{i_2}\binom{k-i_1}{k-i_1-i_2} \\
&= \sum_{i_1=0}^{k} \sum_{i_2=0}^{k} \binom{m}{i_1}\binom{i_1}{i_2}\binom{k-i_1}{k-i_1-i_2} \\
&\vdots \\
&= \sum_{i_1=0}^{k} \cdots \sum_{i_{k-1}=0}^{k} \sum_{i_k=0}^{k-i_1-\cdots-i_{k-1}} \binom{m}{i_1}\binom{i_1}{i_2} \cdots \binom{i_{k-1}}{i_k}\binom{k-i_1-\cdots-i_{k-1}}{k-i_1-\cdots-i_k} \\
&= \sum_{i_k=0}^{k} \cdots \sum_{i_2=0}^{k} \sum_{i_1=0}^{k-i_2-\cdots-i_k} \binom{m}{i_1}\binom{i_1}{i_2} \cdots \binom{i_{k-1}}{i_k}\binom{k-i_1-\cdots-i_{k-1}}{k-i_1-\cdots-i_k}.
\end{aligned}$$

By renaming the indices, we get that

$$\sum_{i_1=0}^{k} \cdots \sum_{i_{k-1}=0}^{k} \sum_{i_k=0}^{k-i_1-\cdots-i_{k-1}} \binom{m}{i_k}\binom{i_k}{i_{k-1}} \cdots \binom{i_2}{i_1}\binom{k-i_2-\cdots-i_k}{k-i_1-\cdots-i_k} = \binom{m+k}{k}. \tag{10}$$

Note that the nonzero terms in the left hand side of (10) will be obtained whenever $i_1 \leqslant \cdots \leqslant i_k$. Since $i_1 \leqslant \cdots \leqslant i_k \leqslant k - i_1 - \cdots - i_{k-1} \leqslant k - (k-1)i_1$, we get that $i_1 \leqslant 1$. If $i_1 = 1$, then $i_1 = \cdots = i_k = 1$ and $\binom{k-i_2-\cdots-i_k}{k-i_1-\cdots-i_k} = 1$. If $i_1 = 0$, then $\binom{k-i_2-\cdots-i_k}{k-i_1-\cdots-i_k} = 1$ again. This completes the proof. □

**Lemma 3.6.** *Let $n \geqslant 2$ and $d, r \geqslant 0$ be three integers with $(n-1)(d-1)+1 \leqslant r \leqslant (n-1)d$. Then,*

$$\sum_{i_1=0}^{d} \cdots \sum_{i_{n-1}=0}^{d} (r - i_1 - \cdots - i_{n-1})\binom{d}{i_1}\binom{i_1}{i_2} \cdots \binom{i_{n-2}}{i_{n-1}} = \left(r - \frac{(n-1)d}{2}\right) n^d \tag{11}$$

*and*

$$\sum_{i_1=0}^{d} \cdots \sum_{i_{n-2}=0}^{d} \sum_{i_{n-1}=r-i_1-\cdots-i_{n-2}}^{d} (r - i_1 - \cdots - i_{n-1})\binom{d}{i_1}\binom{i_1}{i_2} \cdots \binom{i_{n-2}}{i_{n-1}} = -\binom{nd-r}{d+1}. \tag{12}$$

*Proof.* For convenience, let $A$ and $B$ be the left hand sides of (11) and (12), respectively. By substituting $d-i_\ell$ with $i_\ell$ for $\ell=1,\dots,n-1$, we may write

$$\begin{aligned}
A &= \sum_{i_1=0}^{d}\cdots\sum_{i_{n-1}=0}^{d}\big(r-(n-1)d+i_1+\cdots+i_{n-1}\big)\binom{d}{i_1}\binom{d-i_1}{d-i_2}\cdots\binom{d-i_{n-2}}{d-i_{n-1}}\\
&= \sum_{i_1=0}^{d}\cdots\sum_{i_{n-1}=0}^{d}\big(r-(n-1)d+i_1+\cdots+i_{n-1}\big)\binom{d}{i_{n-1}}\binom{i_{n-1}}{i_{n-2}}\cdots\binom{i_2}{i_1}\\
&= \sum_{i_1=0}^{d}\cdots\sum_{i_{n-1}=0}^{d}\big(r-(n-1)d+i_1+\cdots+i_{n-1}\big)\binom{d}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}}\\
&= \big(2r-(n-1)d\big)\sum_{i_1=0}^{d}\cdots\sum_{i_{n-1}=0}^{d}\binom{d}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}}\\
&\quad -\sum_{i_1=0}^{d}\cdots\sum_{i_{n-1}=0}^{d}(r-i_1-\cdots-i_{n-1})\binom{d}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}}\\
&= \big(2r-(n-1)d\big)n^d-A, \qquad (13)
\end{aligned}$$

where the last equality comes from

$$\begin{aligned}
\sum_{i_1=0}^{d}\cdots\sum_{i_{n-1}=0}^{d}\binom{d}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}} &= \sum_{i_1=0}^{d}\sum_{i_2=0}^{i_1}\cdots\sum_{i_{n-1}=0}^{i_{n-2}}\binom{d}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}}\\
&= \sum_{i_1=0}^{d}\sum_{i_2=0}^{i_1}\cdots\sum_{i_{n-2}=0}^{i_{n-3}}\binom{d}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-3}}{i_{n-2}}\sum_{i_{n-1}=0}^{i_{n-2}}\binom{i_{n-2}}{i_{n-1}}\\
&= \sum_{i_1=0}^{d}\sum_{i_2=0}^{i_1}\cdots\sum_{i_{n-3}=0}^{i_{n-4}}\binom{d}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-4}}{i_{n-3}}\sum_{i_{n-2}=0}^{i_{n-3}}\binom{i_{n-3}}{i_{n-2}}2^{i_{n-2}}\\
&\;\;\vdots\\
&= \sum_{i_1=0}^{d}\binom{d}{i_1}(n-1)^{i_1}\\
&= n^d.
\end{aligned}$$

As (11) is deduced from (13), we are done.

We prove (12) by induction on $n$. If $n=2$, then $r=d$ and obviously (12) holds. Let $n\geqslant 3$ and assume that (12) is valid for $n-1$. If $i_1\leqslant d-1$, then the nonzero terms in the left hand side of (12) will be obtained whenever all indices $i_1,\dots,i_{n-1}$ are at most $d-1$. However, $i_{n-1}\geqslant r-i_1-\cdots-i_{n-2}\geqslant (n-1)(d-1)+1-(n-2)(d-1)=d$, a contradiction. Therefore,

$$B=\sum_{i_2=0}^{d}\cdots\sum_{i_{n-2}=0}^{d}\sum_{i_{n-1}=r-d-i_2-\cdots-i_{n-2}}^{d}(r-d-i_2-\cdots-i_{n-1})\binom{d}{i_2}\binom{i_2}{i_3}\cdots\binom{i_{n-2}}{i_{n-1}}. \qquad (14)$$

If $(n-1)(d-1)+2 \leqslant r \leqslant (n-1)d$, then $(n-2)(d-1)+1 \leqslant r-d \leqslant (n-2)d$ and so the right hand side of (14) is equal to $-\binom{(n-1)d-(r-d)}{d+1} = -\binom{nd-r}{d+1}$, by the induction hypothesis. So, it remains to consider the case $r = (n-1)(d-1)+1$. We find from (14) that

$$\begin{aligned} B &= \sum_{i_2=0}^{d} \cdots \sum_{i_{n-2}=0}^{d} \sum_{i_{n-1}=r-d+1-i_2-\cdots-i_{n-2}}^{d} (r-d-i_2-\cdots-i_{n-1}) \binom{d}{i_2}\binom{i_2}{i_3}\cdots\binom{i_{n-2}}{i_{n-1}} \\ &= \sum_{i_2=0}^{d} \cdots \sum_{i_{n-2}=0}^{d} \sum_{i_{n-1}=r-d+1-i_2-\cdots-i_{n-2}}^{d} (r-d+1-i_2-\cdots-i_{n-1}) \binom{d}{i_2}\binom{i_2}{i_3}\cdots\binom{i_{n-2}}{i_{n-1}} \\ &\quad - \sum_{i_2=0}^{d} \cdots \sum_{i_{n-2}=0}^{d} \sum_{i_{n-1}=r-d+1-i_2-\cdots-i_{n-2}}^{d} \binom{d}{i_2}\binom{i_2}{i_3}\cdots\binom{i_{n-2}}{i_{n-1}}. \end{aligned} \tag{15}$$

As $r-d+1 = (n-2)(d-1)+1$, it follows form the induction hypothesis and (15) that

$$\begin{aligned} B &= -\binom{(n-1)d-\big((n-2)(d-1)+1\big)}{d+1} \\ &\quad - \sum_{i_2=0}^{d} \cdots \sum_{i_{n-2}=0}^{d} \sum_{i_{n-1}=r-d+1-i_2-\cdots-i_{n-2}}^{d} \binom{d}{i_2}\binom{i_2}{i_3}\cdots\binom{i_{n-2}}{i_{n-1}} \\ &= -\binom{d+n-3}{d+1} - \sum_{i_2=0}^{d} \cdots \sum_{i_{n-2}=0}^{d} \sum_{i_{n-1}=(n-2)(d-1)+1-i_2-\cdots-i_{n-2}}^{d} \binom{d}{i_2}\binom{i_2}{i_3}\cdots\binom{i_{n-2}}{i_{n-1}}. \end{aligned} \tag{16}$$

We claim that

$$\sum_{i_2=0}^{d} \cdots \sum_{i_{n-2}=0}^{d} \sum_{i_{n-1}=(n-2)(d-1)+1-i_2-\cdots-i_{n-2}}^{d} \binom{d}{i_2}\binom{i_2}{i_3}\cdots\binom{i_{n-2}}{i_{n-1}} = \binom{d+n-3}{d}. \tag{17}$$

The claim trivially holds for $n=3$. So, assume that $n \geqslant 4$. If $i_2 \leqslant d-1$, then the nonzero terms in the left hand side of (17) will be obtained whenever all indices $i_2, \ldots, i_{n-1}$ are at most $d-1$. However, $i_{n-1} \geqslant (n-2)(d-1)+1-i_2-\cdots-i_{n-2} \geqslant (n-2)(d-1)+1-(n-3)(d-1) = d$, a contradiction. Hence, it is enough to show that $C = \binom{d+n-3}{d}$, where

$$C = \sum_{i_3=0}^{d} \cdots \sum_{i_{n-2}=0}^{d} \sum_{i_{n-1}=(n-3)(d-1)-i_3-\cdots-i_{n-2}}^{d} \binom{d}{i_3}\binom{i_3}{i_4}\cdots\binom{i_{n-2}}{i_{n-1}}.$$

By substituting $d-i_\ell$ with $i_\ell$ for $\ell = 3, \ldots, n-1$, we derive that

$$\begin{aligned} C &= \sum_{i_3=0}^{d} \cdots \sum_{i_{n-2}=0}^{d} \sum_{i_{n-1}=0}^{n-3-i_3-\cdots-i_{n-2}} \binom{d}{d-i_3}\binom{d-i_3}{d-i_4}\cdots\binom{d-i_{n-2}}{d-i_{n-1}} \\ &= \sum_{i_3=0}^{n-3} \cdots \sum_{i_{n-2}=0}^{n-3} \sum_{i_{n-1}=0}^{n-3-i_3-\cdots-i_{n-2}} \binom{d}{i_{n-1}}\binom{i_{n-1}}{i_{n-2}}\cdots\binom{i_4}{i_3} \end{aligned}$$

$$= \binom{d+n-3}{d},$$

where the last equality is obtained from Proposition 3.5. This establishes the claim. Now, it follows from (16) and (17) that

$$B = -\binom{d+n-3}{d+1} - \binom{d+n-3}{d} = -\binom{d+n-2}{d+1}.$$

As $r = (n-1)(d-1)+1$, we are done. □

Now, we are ready to prove our main result. Recall Theorem 1.1.

**Theorem 1.1.** *Let $n \geqslant 2$ and $d, r \geqslant 0$ be three integers with $0 \leqslant r \leqslant (n-1)d$. Then,*

$$m_e\left(K_n^d, r\right) = \sum_{i_1=0}^{r-1} \sum_{i_2=0}^{r-i_1-1} \cdots \sum_{i_{n-1}=0}^{r-i_1-\cdots-i_{n-2}-1} (r - i_1 - \cdots - i_{n-1}) \binom{d}{i_1}\binom{i_1}{i_2} \cdots \binom{i_{n-2}}{i_{n-1}}. \tag{18}$$

*Proof.* Fix $n \geqslant 1$ and define $a_n(s,t)$ for every integers $s, t$ with $s \geqslant 1$ and $0 \leqslant t \leqslant s-1$ as follows:

$$a_n(s,t) = \begin{cases} s & \text{if } t = 0. \\ \displaystyle\sum_{i=1}^{\min\{s-t,n-1\}} a_n(s-i, t-1) & \text{otherwise.} \end{cases} \tag{19}$$

We prove by induction on $t$ that

$$a_n(s,t) = \sum_{i=0}^{s-t-1} a_{n-1}(s-t, i)\binom{t}{i} \tag{20}$$

for any $n \geqslant 2$. In view of (19), one concludes that (20) holds for $t = 0$. Let $t \geqslant 1$ and assume that (20) holds for $t-1$. Using (19), we may write

$$\begin{aligned} a_n(s,t) &= \sum_{i=1}^{\min\{s-t,n-1\}} a_n(s-i, t-1) \\ &= \sum_{i=1}^{\min\{s-t,n-1\}} \sum_{j=0}^{s-t-i} a_{n-1}(s-t-i+1, j)\binom{t-1}{j} \\ &= \sum_{j=0}^{s-t-1} a_{n-1}(s-t, j)\binom{t-1}{j} + \sum_{i=2}^{\min\{s-t,n-1\}} \sum_{j=0}^{s-t-i} a_{n-1}(s-t-i+1, j)\binom{t-1}{j} \\ &= \sum_{j=0}^{s-t-1} a_{n-1}(s-t, j)\binom{t-1}{j} + \sum_{j=0}^{s-t-2} \sum_{i=2}^{\min\{s-t-j,n-1\}} a_{n-1}(s-t-i+1, j)\binom{t-1}{j} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{s-t-1} a_{n-1}(s-t,j)\binom{t-1}{j} + \sum_{j=0}^{s-t-2} \sum_{i=1}^{\min\{s-t-j-1,n-2\}} a_{n-1}(s-t-i,j)\binom{t-1}{j} \\
&= \sum_{j=0}^{s-t-1} a_{n-1}(s-t,j)\binom{t-1}{j} + \sum_{j=0}^{s-t-2} a_{n-1}(s-t,j+1)\binom{t-1}{j} \\
&= \sum_{j=0}^{s-t-1} a_{n-1}(s-t,j)\binom{t-1}{j} + \sum_{j=0}^{s-t-1} a_{n-1}(s-t,j)\binom{t-1}{j-1} \\
&= \sum_{j=0}^{s-t-1} a_{n-1}(s-t,j)\binom{t}{j},
\end{aligned}
$$

as required.

By repeatedly using (20), we get that

$$
\begin{aligned}
\sum_{i_1=0}^{r-1} a_n(r,i_1)\binom{d}{i_1} &= \sum_{i_1=0}^{r-1}\sum_{i_2=0}^{r-i_1-1} a_{n-1}(r-i_1,i_2)\binom{d}{i_1}\binom{i_1}{i_2} \\
&\vdots \\
&= \sum_{i_1=0}^{r-1}\sum_{i_2=0}^{r-i_1-1}\cdots\sum_{i_n=0}^{r-i_1-\cdots-i_{n-1}-1} a_1(r-i_1-\cdots-i_{n-1},i_n)\binom{d}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-1}}{i_n} \\
&= \sum_{i_1=0}^{r-1}\sum_{i_2=0}^{r-i_1-1}\cdots\sum_{i_{n-1}=0}^{r-i_1-\cdots-i_{n-2}-1} (r-i_1-\cdots-i_{n-1})\binom{d}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}}, \quad (21)
\end{aligned}
$$

where (21) comes from

$$
a_1(s,t) = \begin{cases} s & \text{if } t=0, \\ 0 & \text{otherwise} \end{cases}
$$

which is in turn obtained from (19).

In order to prove (18) and in view of (21), it is enough to establish that

$$
m_e\left(K_n^d, r\right) = \sum_{j=0}^{r-1} a_n(r,j)\binom{d}{j} \qquad (22)
$$

when $0 \leqslant r \leqslant (n-1)d$. We prove (22) by induction on $d$. If $d=0$, then $r=0$ and (22) clearly holds. Let $d \geqslant 1$ and assume that (22) holds for $d-1$. First, assume that $r \leqslant (n-1)(d-1)$. We have $g=1$ and so it follows from (8) that

$$
\begin{aligned}
m_e\left(K_n^d, r\right) &= \sum_{i=0}^{n-1} m_e\left(K_n^{d-1}, r-i\right) \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{r-i-1} a_n(r-i,j)\binom{d-1}{j}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=0}^{r-1} a_n(r,j)\binom{d-1}{j} + \sum_{i=1}^{n-1}\sum_{j=0}^{r-i-1} a_n(r-i,j)\binom{d-1}{j} \\
&= \sum_{j=0}^{r-1} a_n(r,j)\binom{d-1}{j} + \sum_{j=0}^{r-2}\sum_{i=1}^{\min\{r-j-1,n-1\}} a_n(r-i,j)\binom{d-1}{j} \\
&= \sum_{j=0}^{r-1} a_n(r,j)\binom{d-1}{j} + \sum_{j=0}^{r-2} a_n(r,j+1)\binom{d-1}{j} \\
&= \sum_{j=0}^{r-1} a_n(r,j)\binom{d-1}{j} + \sum_{j=0}^{r-1} a_n(r,j)\binom{d-1}{j-1} \\
&= \sum_{j=0}^{r-1} a_n(r,j)\binom{d}{j},
\end{aligned}
$$

as required. Next, assume that $(n-1)(d-1)+1 \leqslant r \leqslant (n-1)d$. In order to prove (22) and in view of (21), it suffices to show that the right hand sides of (9) and (18) are equal. Equivalently, by letting

$$
S = \sum_{i_1=0}^{r-1}\sum_{i_2=0}^{r-i_1-1}\cdots\sum_{i_{n-1}=0}^{r-i_1-\cdots-i_{n-2}-1}(r-i_1-\cdots-i_{n-1})\binom{d}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}},
$$

we should show that

$$
S = \binom{nd-r}{d+1} + \left(r - \frac{(n-1)d}{2}\right)n^d.
$$

We claim that

$$
S = \sum_{i_1=0}^{d}\cdots\sum_{i_{n-2}=0}^{d}\sum_{i_{n-1}=0}^{r-i_1-\cdots-i_{n-2}-1}(r-i_1-\cdots-i_{n-1})\binom{d}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}}.
$$

To see this, assume that $1 \leqslant k \leqslant n-2$ and $s = r - i_1 - \cdots - i_{k-1} - 1$. The upper bound of $k$th summation notation in the right hand side of (18) is $s$. We show that $s$ can be replaced by $d$. Note that $\binom{d}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}} = 0$ if $i_k > d$. Therefore, there is nothing to prove if $s \geqslant d$. So, assume that $s \leqslant d-1$. Since the upper bound of $(k+1)$th summation notation in the right hand side of (18) is $s - i_k$, the upper bound of $k$th summation notation can be go up to $d$ instead of $s$. This establishes the claim.

Now, it follows from Lemma 3.6 that

$$
\begin{aligned}
S &= \sum_{i_1=0}^{d}\cdots\sum_{i_{n-1}=0}^{d}(r-i_1-\cdots-i_{n-1})\binom{d}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}} \\
&\quad - \sum_{i_1=0}^{d}\cdots\sum_{i_{n-2}=0}^{d}\sum_{i_{n-1}=r-i_1-\cdots-i_{n-2}}^{d}(r-i_1-\cdots-i_{n-1})\binom{d}{i_1}\binom{i_1}{i_2}\cdots\binom{i_{n-2}}{i_{n-1}} \\
&= \left(r - \frac{(n-1)d}{2}\right)n^d + \binom{nd-r}{d+1}.
\end{aligned}
$$

This establishes (22) and completes the proof. □

**Remark 3.7.** The explicit value of $m_e(K_2^d, r)$ was presented in [11, 13] by a rather complicated formula. A simple expression for $m_e(K_2^d, r)$ is obtained from (18). Indeed,

$$m_e\left(K_2^d, r\right) = \sum_{i=0}^{r} (r-i)\binom{d}{i}$$

for any integer $r$ with $0 \leqslant r \leqslant d$.

Recall that the *weight* of a tuple is defined to be the number of its nonzero components. The following result was proved for $n = 2$ in [13]. Recall Theorem 1.2.

**Theorem 1.2.** *Let $n \geqslant 2$ and $r$ be two fixed positive integers and let $d$ be an integer tending to infinity. Then,*

$$m\left(K_n^d, r\right) = \big(1 + o(1)\big)\frac{d^{r-1}}{r!}.$$

*Proof.* It follows from (18) that $m_e(K_n^d, r) \geqslant (1 + o(1))\binom{d}{r-1}$ and so we obtain from (1) that

$$m\left(K_n^d, r\right) \geqslant \frac{1 + o(1)}{r}\binom{d}{r-1}. \tag{23}$$

Consider a family $\mathscr{F}$ of $r$-subsets of $[\![d]\!]$ such that every $(r-1)$-subset of $[\![d]\!]$ is contained in at least one element of $\mathscr{F}$. Let $U \subseteq [\![2]\!]^d$ be the set of all characteristic vectors corresponding to elements of $\mathscr{F}$ and let $W \subseteq V(K_n^d)$ be the set of all vertices of weight $r-2$. We claim that $U \cup W$ percolates in the $r$-neighbor bootstrap percolation process on $K_n^d$.

First, note that the vertices of weights $r-3, \ldots, 1, 0$ can be respectively activated. This is possible since every vertex of weight $r-i$ is adjacent to $(d-r+i)(n-1) \geqslant r$ vertices of weight $r-i+1$ for $i = 3, \ldots, r$. Next, note that the vertices of weight $r-1$ can be activated. To see this, we show that the vertices of weight $r-1$ with $0, 1, \ldots, r-1$ components in $[\![n]\!] \setminus [\![2]\!]$ can be respectively activated. All vertices of weight $r-1$ whose all components are in $[\![2]\!]$ can be activated, since such vertices have $r-1$ neighbors in $W$ and at least one neighbor in $U$. Now, for $i = 1, \ldots, r-1$, every vertex of weight $r-1$ with $i$ components in $[\![n]\!] \setminus [\![2]\!]$ can be activated, since such vertices have $r-1$ neighbors in $W$ and at least one neighbor of weight $r-1$ with $i-1$ components in $[\![n]\!] \setminus [\![2]\!]$. Finally, note that the vertices of weights $r, \ldots, d$ can be respectively activated. This is possible since every vertex of weight $r+i$ is adjacent to $r+i$ vertices of weight $r+i-1$ for $i = 0, 1, \ldots, d-r$. This establishes the claim.

By a result of Rödl [16], there exists such a family $\mathscr{F}$ with $|\mathscr{F}| = \frac{1+o(1)}{r}\binom{d}{r-1}$. Therefore,

$$m\left(K_n^d, r\right) \leqslant |U| + |W| = \frac{1 + o(1)}{r}\binom{d}{r-1} + (n-1)^{r-2}\binom{d}{r-2}. \tag{24}$$

The result follows from (23) and (24). □

It is worth mentioning that it remains as an open challenging problem to find the exact formula for $m(K_n^d, r)$ in general cases. We refer to see [6] for some results on small $r$.

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