All triples for orthogonal designs of order 40

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Abstract

The use of amicable sets of eight circulant matrices and four negacyclic matrices to generate orthogonal designs is relatively new. We find all 1841 possible orthogonal designs of order 40 in three variables, using only these new techniques.

Keywords: Orthogonal designs, amicable sets of matrices, negacyclic matrices.

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1 Introduction

An *orthogonal design* A, of order n, and type $(s_1, s_2, \ldots, s_u)$, in the commuting variables $\pm x_1, \pm x_2, \ldots, \pm x_u$, denoted $OD(n; s_1, s_2, \ldots, s_u)$ is a square matrix of order n with entries $\pm x_k$ or 0, where each x_k occurs s_k times in each row and column such that the distinct rows are pairwise orthogonal. In other words,

$$AA^T = (s_1 x_1^2 + \cdots + s_u x_u^2) I_n,$$

where I_n is the identity matrix. It is known that the maximum number of variables in an orthogonal design is $\rho(n)$, the Radon number, defined as $\rho(n) = 8c + 2^d$, where $n = 2^a b$, b odd, and $a = 4c + d, 0 \le d < 4$.

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In [4] it is shown that all full 3-tuples (s_1, s_2, s_3), $s_1 + s_2 + s_3 = 40$, are types of orthogonal designs of order 40. It is conjectured that all possible 3-tuples are types of an orthogonal design of order $8n$, see [2]. This conjecture has been verified for $n = 1, 2, 3, 4, 6$ [2, 6]. In this paper, we will show that the conjecture also holds for $n = 5$. While the use of circulant matrices in constructing orthogonal designs is standard, we have shown in a previous paper [6] that negacyclic matrices may be used for those cases where the circulant matrices either do not exist or are hard to find. Our approach in this paper is to first generate, from all possible sets of four complementary circulant matrices of order five in four variables, all possible sets of amicable matrices in eight or less variables. In doing so, we are able to construct a total of 1785 triples out of 1841 possible triples. For the remaining 56 triples, we use a simple algorithm using negacyclic matrices.

2 Amicable sets of matrices

Two orthogonal designs X and Y are called *amicable* if $XY^t = YX^t$. A set $\{A_1, A_2, \ldots, A_{2n}\}$ of square real matrices is said to be *amicable* if

$$\sum_{i=1}^{n} (A_{\sigma(2i-1)} A^t_{\sigma(2i)} - A_{\sigma(2i)} A^t_{\sigma(2i-1)}) = 0,$$

for some permutation σ of the set $\{1, 2, \ldots, 2n\}$. We say that $\{A_{\sigma(2i-1)}\}_{i=1,\ldots,n}$ *matches* with $\{A_{\sigma(2i)}\}_{i=1,\ldots,n}$. For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. A set of matrices $\{A_1, A_2, \ldots, A_n\}$ is said to satisfy an additive property, if the matrix $\sum_{i=1}^{n} AA_i^t$ is a multiple of the identity matrix.

An amicable set of eight circulant matrices $\{A_1, A_2, \ldots, A_8\}$ of order n satisfying an additive property can be used in the Kharaghani array [9]:

$$K = \begin{pmatrix}
A_1 & A_2 & A_4R & A_3R & A_6R & A_5R & A_8R & A_7R \\
-A_2 & A_1 & A_3R & -A_4R & A_5R & -A_6R & A_7R & -A_8R \\
-A_4R & -A_3R & A_1 & A_2 & -A_8^tR & A_7^tR & A_6^tR & -A_5^tR \\
-A_3R & A_4R & -A_2 & A_1 & A_7^tR & A_8^tR & -A_5^tR & -A_6^tR \\
-A_6R & -A_5R & A_8^tR & -A_7^tR & A_1 & A_2 & -A_4^tR & A_3^tR \\
-A_5R & A_6R & -A_7^tR & -A_8^tR & -A_2 & A_1 & A_3^tR & A_4^tR \\
-A_8R & -A_7R & -A_6^tR & A_5^tR & A_4^tR & -A_3^tR & A_1 & A_2 \\
-A_7R & A_8R & A_5^tR & A_6^tR & -A_3^tR & -A_4^tR & -A_2 & A_1
\end{pmatrix},$$

where R is the back–identity matrix of order n, to obtain an orthogonal matrix of order $8n$.

We began our search from the known sets of four circulant matrices of order 5 in 4 variables mentioned in [2, pages 349–351]. These are sets of four matrices $\{A, B, C, D\}$ in four variables $\{a, b, c, d\}$ satisfying an additive property. We then changed the four variables to new variables thereby obtaining eight circulant matrices $\{A, B, C, D, E, F, G, H\}$ in eight variables $\{a, b, c, d, e, f, g, h\}$ satisfying an additive property. We did a complete search for all possible amicable sets of eight matrices obtained from these matrices. A complete search includes collapsing of variables [4] and equating some variables to zero. Certain triples could only be obtained by zeroing of some variables in the known sets of four circulant matrices of order 5, prior to collapsing and matching. These triples are listed in Table 1.

An amicable set obtained where the matching is between $\{A, B, C, D\}$ and $\{E, F, G, H\}$, is called a *special amicable set*, [5]. Each of the special amicable sets of matrices can be used in other arrays to generate infinitely many orthogonal designs, see [5, 9]. It is this nice property of amicable sets that was the reason why we searched for as many as possible amicable sets of matrices in the first instance. More than 1350 of the amicable sets of matrices found were special.

The following theorem is obtained by using amicable sets found in the above searches, including those listed in Table 1, in the Kharaghani array, K.

Theorem 1 *All triples are types of orthogonal designs of order* 40 *constructible from eight circulant matrices via the Kharaghani array, K, except possibly those in Table 2.*

Table 1: Amicable sets for some OD's of order 40 in 3 variables

type	A_1 / A_2	A_3 / A_4	A_5 / A_6	A_7 / A_8
$(5, 11, 20)$	$(a, c, c, \bar{c}, \bar{c})$	$(0, b, a, a, \bar{b})$	$(a, \bar{a}, b, b, b)$	$(b, c, b, \bar{b}, 0)$
	$(0, c, \bar{c}, c, \bar{c})$	$(c, c, c, c, \bar{c})$	$(c, c, c, \bar{c}, c)$	$(b, \bar{c}, b, \bar{b}, 0)$
$(6, 6, 25)$	$(c, c, \bar{c}, \bar{c}, c)$	$(c, c, c, c, \bar{c})$	$(c, b, a, \bar{a}, b)$	$(b, a, a, \bar{b}, c)$
	$(0, a, \bar{c}, b, \bar{c})$	$(c, \bar{b}, \bar{c}, 0, a)$	$(c, c, c, \bar{c}, c)$	$(c, \bar{c}, c, \bar{c}, 0)$
$(7, 9, 20)$	$(a, c, c, \bar{c}, \bar{c})$	$(a, a, \bar{b}, 0, b)$	$(a, \bar{a}, b, b, b)$	$(a, c, b, \bar{b}, 0)$
	$(0, c, \bar{c}, c, \bar{c})$	$(c, c, c, c, \bar{c})$	$(\bar{c}, c, c, c, c)$	$(a, \bar{c}, b, \bar{b}, 0)$
$(7, 10, 19)$	$(c, \bar{c}, \bar{c}, c, \bar{c})$	$(\bar{a}, 0, b, b, \bar{a})$	$(c, \bar{c}, c, c, c)$	$(0, c, c, \bar{c}, \bar{c})$
	$(0, b, b, \bar{b}, a)$	$(c, c, \bar{c}, c, c)$	$(0, a, \bar{a}, b, a)$	$(a, b, b, \bar{b}, b)$
$(8, 8, 21)$	$(c, c, \bar{c}, c, \bar{c})$	$(c, b, \bar{a}, b, a)$	$(c, \bar{b}, a, \bar{b}, \bar{a})$	$(\bar{a}, \bar{b}, \bar{a}, b, 0)$
	$(c, \bar{c}, \bar{c}, 0, c)$	$(c, c, c, c, \bar{c})$	$(c, c, c, c, \bar{c})$	$(a, b, a, \bar{b}, 0)$
$(10, 11, 15)$	$(a, a, \bar{b}, \bar{b}, b)$	$(c, c, \bar{c}, 0, c)$	$(a, b, \bar{a}, b, b)$	$(b, \bar{a}, a, \bar{b}, 0)$
	$(\bar{c}, 0, a, \bar{c}, c)$	$(a, b, a, b, \bar{b})$	$(c, \bar{c}, 0, a, c)$	$(c, \bar{c}, c, c, c)$
$(11, 11, 11)$	$(a, a, a, \bar{a}, \bar{a})$	$(b, 0, 0, \bar{b}, 0)$	$(0, b, b, b, 0)$	$(b, \bar{c}, b, \bar{b}, 0)$
	$(c, \bar{c}, c, c, c)$	$(a, \bar{a}, a, c, \bar{c})$	$(a, c, a, a, \bar{c})$	$(b, c, b, \bar{b}, 0)$

3 Construction using negacyclic matrices

A negacyclic matrix of order n is any polynomial in the negashift matrix U of order n, where $U = [u_{ij}]$, $u_{i(i+1)} = 1$, $i = 1, 2, \cdots, n-1$, $u_{n1} = -1$, and $u_{ij} = 0$ if $j - i \not\equiv 1 \bmod n$. The class of negacyclic matrices of order n form a commutative subring of the ring of all matrices of order n. For more details see [1, 7, 8].

It is known that if A_1, A_2, A_3, A_4 are four negacyclic matrices of order n satisfying an additive property, then upon substitution the Goethals-Seidel array

$$G = \begin{pmatrix} A_1 & A_2R & A_3R & A_4R \\ -A_2R & A_1 & A_4^tR & -A_3^tR \\ -A_3R & -A_4^tR & A_1 & A_2^tR \\ -A_4R & A_3^tR & -A_2^tR & A_1 \end{pmatrix},$$

becomes an orthogonal matrix of order $4n$. Note that the matrix R is the back–diagonal identity of order n.

It is not hard to see that not every orthogonal design of order 40 is constructible from four or eight circulant matrices of order 10 or 5, respectively. The techniques to show this are well known and require some computer computations. However, our aim in this note is to show the usefulness of negacyclic matrices. To this end, we did a *complete search* for amicable sets of eight matrices and this resulted in 1785 triples of order 40. This is all triples that can be found using eight circulant matrices in K, the Kharaghani array. There remained 56 more triples to be found. We searched next using negacyclic matrices of order 10 and found all of the remaining orthogonal designs. Our search consisted of three steps. In the first step we generated and saved all possible negacyclic $(0, \pm 1)$-matrices of order 10 and classified them by looking at products of pairs of rows. In the second step, for any $1 \leq s \leq 40$, we found all weighing matrices $W(40, s)$ constructible from 4 negacyclic matrices via G, the Goethals-Seidel array. To retain a smaller number of weighing matrices, we saved only those for which one of the four negacyclic matrices was skew. In the last step for finding an $OD(40, s_1, s_2, s_3)$, we searched for disjoint weighing matrices $W(40, s_1)$, $W(40, s_2)$, and $W(40, s_3)$ among those found in the second step and tested if their sum gives an orthogonal design. It is notable that it took only a few seconds on a single desktop computer to execute our program to find any one of the 56 orthogonal designs listed in Table 2. We have the following theorem.

Theorem 2 *All* 1841 *possible* $OD(40; s_1, s_2, s_3)$ *are constructible from either eight circulant matrices of order* 5*, or four negacyclic matrices of order* 10*.*

The complete set of 1841 orthogonal designs of order 40 in three variables is available at the website http://www.cs.uleth.ca/OD40triples. There we also list either the first row of eight amicable or special amicable circulant matrices of order 5, or the first row of four negacyclic matrices of order 10 used in the construction of each orthogonal design.

Table 2: First rows of four sets of complementary negacyclic matrices for given OD's of order 40 in 3 variables

type	A_1	A_2
	A_3	A_4
$(1, 15, 23)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{b}, \bar{c}, c, \bar{b}, \bar{c}, c, c, c, \bar{c}, \bar{b})$
	$(\bar{b}, \bar{b}, c, \bar{c}, b, \bar{b}, b, \bar{b}, \bar{b}, \bar{c})$	$(\bar{a}, \bar{b}, c, \bar{c}, \bar{b}, \bar{b}, \bar{b}, \bar{c}, c, \bar{b})$
$(2, 4, 33)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{b}, \bar{c}, c, c, \bar{c}, \bar{b}, \bar{c}, \bar{c}, \bar{c}, \bar{c})$
	$(\bar{b}, c, \bar{c}, \bar{c}, c, \bar{b}, \bar{c}, c, c, \bar{c})$	$(\bar{a}, \bar{c}, c, \bar{c}, c, \bar{a}, c, \bar{c}, \bar{c}, c)$
$(2, 7, 30)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{a}, \bar{c}, c, \bar{b}, c, c, c, \bar{c}, c, \bar{c})$
	$(\bar{b}, \bar{b}, c, \bar{c}, \bar{c}, \bar{b}, c, \bar{c}, c, c)$	$(\bar{a}, \bar{c}, \bar{b}, b, c, \bar{c}, c, c, \bar{b}, \bar{c})$
$(2, 13, 24)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{b}, \bar{b}, \bar{c}, \bar{c}, \bar{c}, c, \bar{b}, b, \bar{b}, \bar{b})$
	$(\bar{a}, \bar{c}, c, \bar{b}, \bar{b}, \bar{b}, c, \bar{c}, c, \bar{c})$	$(\bar{a}, \bar{c}, \bar{b}, b, c, \bar{c}, \bar{b}, c, \bar{b}, \bar{c})$
$(2, 16, 21)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{b}, \bar{b}, \bar{c}, \bar{c}, c, \bar{c}, c, b, \bar{b}, \bar{b})$
	$(\bar{a}, \bar{b}, c, \bar{c}, b, b, b, \bar{b}, b, \bar{b})$	$(\bar{a}, c, \bar{c}, \bar{b}, \bar{b}, c, \bar{b}, \bar{c}, \bar{b}, c)$
$(3, 5, 31)$	$(0, \bar{c}, \bar{c}, \bar{b}, \bar{c}, c, \bar{c}, \bar{b}, \bar{c}, \bar{c})$	$(\bar{a}, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, c, c, \bar{c})$
	$(\bar{b}, c, \bar{b}, c, \bar{c}, c, c, c, \bar{b}, \bar{c})$	$(\bar{a}, \bar{c}, c, c, c, \bar{a}, \bar{c}, c, c, \bar{c})$
$(3, 6, 30)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{b}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{b}, c, \bar{c}, c)$
	$(\bar{a}, \bar{c}, \bar{b}, c, c, \bar{c}, c, \bar{c}, b, \bar{c})$	$(\bar{a}, \bar{c}, \bar{c}, b, c, \bar{a}, \bar{c}, c, \bar{b}, \bar{c})$
$(3, 7, 29)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{a}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c})$
	$(\bar{a}, \bar{b}, \bar{c}, c, b, \bar{a}, c, \bar{c}, \bar{c}, c)$	$(\bar{b}, \bar{b}, c, \bar{c}, \bar{b}, \bar{c}, c, \bar{b}, c, \bar{b})$
$(3, 8, 28)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{b}, \bar{c}, c, \bar{b}, \bar{c}, c, \bar{c}, c, c, c)$
	$(\bar{a}, \bar{c}, \bar{c}, c, c, \bar{a}, \bar{b}, \bar{b}, c, \bar{c})$	$(\bar{a}, \bar{c}, b, \bar{b}, c, \bar{c}, b, c, \bar{c}, \bar{b})$
$(3, 11, 22)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{b}, 0, \bar{b}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{a}, c, \bar{c}, \bar{b}, c, \bar{b}, c, \bar{b}, c, c)$
	$(\bar{a}, \bar{c}, c, 0, c, \bar{a}, \bar{c}, c, 0, \bar{c})$	$(\bar{b}, \bar{b}, \bar{c}, c, b, c, \bar{c}, b, \bar{b}, \bar{b})$
$(3, 11, 25)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{b}, \bar{b}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{b})$
	$(\bar{a}, \bar{b}, \bar{b}, \bar{c}, c, \bar{b}, c, \bar{b}, \bar{c}, \bar{b})$	$(\bar{a}, c, \bar{b}, b, \bar{c}, \bar{a}, c, \bar{c}, \bar{b}, c)$
$(3, 12, 24)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{a}, \bar{c}, \bar{c}, c, c, \bar{a}, \bar{b}, \bar{b}, c, \bar{c})$
	$(\bar{b}, \bar{b}, \bar{c}, c, \bar{b}, \bar{c}, \bar{b}, \bar{c}, c, \bar{b})$	$(\bar{a}, \bar{c}, c, \bar{c}, \bar{b}, b, \bar{c}, b, \bar{b}, \bar{b})$
$(3, 13, 23)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{b}, \bar{b}, \bar{b}, \bar{c}, \bar{b}, b, b, \bar{c}, \bar{c}, c)$
	$(\bar{a}, c, \bar{c}, c, \bar{b}, b, \bar{b}, \bar{c}, c, c)$	$(\bar{a}, \bar{b}, c, \bar{c}, b, \bar{a}, \bar{b}, c, \bar{c}, \bar{b})$
$(3, 14, 22)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{b}, \bar{c}, \bar{c}, c, \bar{b}, c, \bar{c}, b, c, \bar{b})$
	$(\bar{a}, \bar{b}, \bar{c}, c, b, \bar{a}, b, \bar{b}, \bar{b}, b)$	$(\bar{a}, c, \bar{c}, \bar{b}, \bar{b}, c, \bar{b}, \bar{b}, \bar{c}, c)$
$(3, 16, 20)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{b}, \bar{b}, \bar{c}, c, c, \bar{c}, \bar{b}, b, \bar{b}, \bar{c})$
	$(\bar{a}, \bar{b}, c, \bar{c}, \bar{b}, \bar{b}, \bar{b}, \bar{c}, c, \bar{b})$	$(\bar{a}, \bar{b}, c, \bar{c}, b, \bar{a}, b, \bar{b}, \bar{b}, b)$
$(4, 4, 31)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, c, c, \bar{c}, \bar{c})$
	$(\bar{b}, \bar{b}, \bar{c}, c, c, \bar{a}, a, c, c, \bar{c})$	$(\bar{b}, \bar{c}, c, \bar{c}, a, a, c, \bar{c}, c, \bar{b})$
$(4, 5, 30)$	$(0, \bar{c}, \bar{c}, \bar{b}, \bar{c}, c, \bar{c}, \bar{b}, \bar{c}, \bar{c})$	$(\bar{a}, \bar{c}, c, c, \bar{c}, \bar{a}, \bar{c}, \bar{c}, \bar{c}, \bar{c})$
	$(\bar{b}, c, \bar{b}, c, \bar{c}, c, c, c, \bar{b}, \bar{c})$	$(\bar{a}, c, \bar{c}, \bar{c}, c, \bar{a}, \bar{c}, c, c, \bar{c})$
$(4, 6, 29)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{a}, \bar{c}, \bar{b}, \bar{b}, c, \bar{c}, c, c, c, \bar{a})$
	$(\bar{a}, \bar{a}, c, \bar{c}, c, \bar{c}, \bar{c}, b, \bar{b}, c)$	$(\bar{b}, \bar{c}, c, c, \bar{c}, \bar{b}, \bar{c}, c, c, \bar{c})$
$(4, 7, 28)$	$(0, \bar{c}, \bar{c}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, \bar{c})$	$(\bar{a}, \bar{c}, \bar{c}, c, \bar{c}, \bar{c}, \bar{c}, c, c, \bar{a})$
	$(\bar{a}, \bar{a}, \bar{b}, \bar{c}, c, c, \bar{c}, c, \bar{c}, \bar{b})$	$(\bar{b}, \bar{b}, c, \bar{b}, \bar{c}, c, \bar{c}, b, \bar{b}, c)$

Table 2: Continued

type	A_1 A_3	A_2 A_4
$(4,8,27)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{b},\overline{c},\overline{c},\overline{c},c,c,\overline{c},\overline{b},c,\overline{c})$
	$(\overline{a},\overline{b},\overline{b},b,\overline{c},\overline{a},\overline{c},c,\overline{c},\overline{c})$	$(\overline{a},c,\overline{c},c,b,\overline{a},b,\overline{c},c,b)$
$(4,10,25)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{b},\overline{c},c,\overline{b},\overline{c},c,c,c,\overline{c},c)$
	$(\overline{a},\overline{a},\overline{c},\overline{b},c,\overline{b},b,c,\overline{b},\overline{c})$	$(\overline{a},\overline{c},c,\overline{b},b,b,b,\overline{c},c,\overline{a})$
$(4,11,24)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{b},\overline{c},c,\overline{b},\overline{c},c,c,c,\overline{c},\overline{b})$
	$(\overline{a},c,\overline{b},\overline{c},\overline{b},\overline{b},\overline{b},b,\overline{c},\overline{a})$	$(\overline{a},\overline{a},\overline{b},c,b,\overline{c},c,\overline{c},c,\overline{b})$
$(4,12,23)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{b},\overline{b},b,\overline{c},\overline{c},c,b,\overline{b},c,\overline{b})$
	$(\overline{a},\overline{b},\overline{c},c,c,\overline{a},\overline{c},c,b,b)$	$(\overline{a},\overline{c},c,\overline{c},b,\overline{a},c,\overline{c},\overline{b},\overline{b})$
$(4,13,22)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{b},\overline{b},\overline{c},c,b,\overline{c},b,\overline{c},c,\overline{b})$
	$(\overline{a},c,\overline{b},\overline{c},c,\overline{a},b,b,c,b)$	$(\overline{a},\overline{c},c,b,\overline{c},\overline{a},\overline{b},c,b,\overline{b})$
$(4,14,21)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{b},\overline{b},\overline{c},c,c,\overline{b},\overline{c},c,\overline{b},\overline{c})$
	$(\overline{a},\overline{b},b,\overline{b},\overline{b},\overline{b},\overline{c},\overline{b},b,\overline{a})$	$(\overline{a},\overline{a},\overline{b},c,b,\overline{c},c,\overline{c},c,\overline{b})$
$(5,6,28)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{c},\overline{c},\overline{c},c,\overline{c},c,\overline{c},\overline{c},c,\overline{c})$
	$(\overline{a},\overline{a},\overline{b},\overline{c},c,\overline{b},c,c,b,\overline{a})$	$(\overline{a},\overline{c},\overline{a},c,b,\overline{c},\overline{b},c,\overline{b},\overline{c})$
$(5,7,25)$	$(0,\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c})$	$(\overline{b},\overline{c},c,\overline{c},c,\overline{b},\overline{c},\overline{c},\overline{c},\overline{c})$
	$(\overline{a},\overline{a},\overline{b},\overline{c},c,\overline{b},c,c,b,\overline{a})$	$(\overline{a},0,\overline{a},\overline{c},b,c,0,c,\overline{b},\overline{c})$
$(5,7,27)$	$(0,\overline{c},\overline{c},\overline{a},\overline{c},c,\overline{c},\overline{a},\overline{c},\overline{c})$	$(\overline{a},c,\overline{a},c,\overline{c},c,c,c,\overline{a},\overline{c})$
	$(\overline{b},\overline{c},\overline{b},c,\overline{c},\overline{b},b,\overline{c},\overline{c},\overline{c})$	$(\overline{b},\overline{b},\overline{c},\overline{c},c,\overline{c},\overline{c},c,\overline{b},\overline{c})$
$(5,9,25)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{b},\overline{c},c,\overline{c},\overline{c},c,b,\overline{b},\overline{c},\overline{c})$
	$(\overline{a},c,\overline{c},\overline{a},\overline{b},c,\overline{c},\overline{a},b,\overline{c})$	$(\overline{a},\overline{c},b,c,\overline{c},b,\overline{a},c,b,b)$
$(5,12,22)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{a},\overline{c},\overline{a},c,\overline{b},\overline{c},\overline{c},b,c,\overline{c})$
	$(\overline{b},\overline{b},\overline{b},c,\overline{c},b,\overline{b},\overline{c},b,\overline{b})$	$(\overline{a},\overline{a},b,c,\overline{c},c,\overline{c},\overline{b},\overline{b},\overline{a})$
$(5,13,21)$	$(0,\overline{c},\overline{c},\overline{b},\overline{c},c,\overline{c},\overline{b},\overline{c},\overline{c})$	$(\overline{b},c,\overline{b},\overline{c},c,\overline{b},b,\overline{c},\overline{c},\overline{c})$
	$(\overline{a},\overline{b},\overline{c},c,\overline{b},\overline{c},\overline{a},c,\overline{c},\overline{c})$	$(\overline{a},\overline{c},b,\overline{a},b,c,\overline{b},\overline{a},b,b)$
$(6,6,27)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{b},\overline{c},\overline{a},c,\overline{c},c,c,\overline{b},\overline{a},c)$
	$(\overline{b},c,\overline{a},\overline{c},\overline{c},\overline{b},c,\overline{a},\overline{c},c)$	$(\overline{b},\overline{c},a,\overline{b},c,\overline{c},c,\overline{c},\overline{a},\overline{c})$
$(6,7,26)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{b},\overline{c},\overline{b},c,\overline{c},\overline{b},c,c,c,\overline{c})$
	$(\overline{a},\overline{a},\overline{c},\overline{b},c,\overline{c},b,\overline{c},\overline{b},\overline{a})$	$(\overline{a},c,\overline{a},\overline{c},c,\overline{c},\overline{a},b,c,\overline{c})$
$(6,8,25)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{b},\overline{c},c,\overline{b},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{b})$
	$(\overline{a},\overline{c},\overline{b},c,\overline{c},a,\overline{a},b,b,\overline{c})$	$(\overline{a},\overline{a},c,b,\overline{c},\overline{a},\overline{b},c,\overline{c},c)$
$(6,9,24)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{b},\overline{c},\overline{b},c,c,\overline{c},c,b,\overline{b},c)$
	$(\overline{a},\overline{c},\overline{b},c,\overline{c},a,\overline{a},b,b,\overline{c})$	$(\overline{a},\overline{a},c,b,\overline{c},\overline{a},\overline{b},c,\overline{c},c)$
$(6,10,23)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{a},\overline{c},\overline{b},\overline{b},c,\overline{b},\overline{c},c,c,\overline{a})$
	$(\overline{a},\overline{b},c,\overline{c},b,\overline{a},\overline{b},\overline{c},c,\overline{b})$	$(\overline{a},\overline{a},\overline{c},c,\overline{c},b,c,b,\overline{b},\overline{c})$
$(6,11,22)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{a},\overline{b},\overline{a},b,\overline{b},\overline{c},c,\overline{a},\overline{c},\overline{c})$
	$(\overline{a},c,\overline{c},\overline{a},\overline{c},c,b,b,\overline{a},b)$	$(\overline{b},\overline{b},c,\overline{c},b,c,b,\overline{c},c,\overline{b})$
$(6,12,21)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{b},\overline{c},\overline{b},c,\overline{c},\overline{b},c,\overline{b},\overline{c},\overline{c})$
	$(\overline{a},\overline{c},c,\overline{c},\overline{b},\overline{b},c,\overline{c},c,\overline{a})$	$(\overline{a},\overline{a},\overline{b},\overline{a},b,\overline{b},b,b,\overline{a},\overline{b})$

Table 2: Continued

type	A_1 / A_3	A_2 / A_4
$(6,13,20)$	$(0,\overline{c},\overline{c},\overline{b},\overline{c},c,\overline{c},\overline{b},\overline{c},\overline{c})$	$(\overline{a},\overline{c},\overline{b},\overline{c},b,c,\overline{a},\overline{c},c,c)$
	$(\overline{a},c,\overline{b},c,\overline{a},c,\overline{b},\overline{c},\overline{b},c)$	$(\overline{a},\overline{b},b,\overline{c},b,\overline{a},\overline{b},c,\overline{b},\overline{b})$
$(6,16,17)$	$(0,\overline{c},\overline{c},\overline{b},\overline{c},b,\overline{c},\overline{b},\overline{c},\overline{c})$	$(\overline{a},\overline{b},\overline{c},\overline{c},c,a,\overline{a},b,\overline{c},b)$
	$(\overline{b},\overline{b},\overline{b},c,b,\overline{c},\overline{c},c,\overline{c},\overline{b})$	$(\overline{a},\overline{a},c,\overline{c},b,\overline{a},b,b,\overline{b},b)$
$(7,7,23)$	$(\overline{c},\overline{c},\overline{c},\overline{c},\overline{c},\overline{c},0,c,\overline{c},\overline{c})$	$(\overline{b},\overline{c},\overline{c},c,c,\overline{a},\overline{c},c,\overline{c},c)$
	$(\overline{b},\overline{b},c,0,a,\overline{b},\overline{c},c,\overline{a},a)$	$(\overline{b},c,\overline{c},\overline{a},\overline{a},b,\overline{b},c,0,a)$
$(7,8,24)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{b},\overline{b},\overline{c},c,c,\overline{b},b,c,c,\overline{c})$
	$(\overline{a},\overline{c},c,\overline{b},b,c,\overline{c},\overline{a},\overline{b},\overline{b})$	$(\overline{a},\overline{a},c,\overline{a},\overline{c},c,\overline{c},a,\overline{a},c)$
$(7,9,23)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{a},\overline{a},\overline{c},c,\overline{c},\overline{b},\overline{a},b,\overline{c},\overline{c})$
	$(\overline{a},\overline{b},\overline{b},\overline{c},c,b,\overline{b},c,\overline{c},\overline{a})$	$(\overline{a},c,\overline{c},b,\overline{a},c,b,\overline{c},b,c)$
$(7,10,22)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{a},\overline{c},\overline{a},\overline{b},b,\overline{a},\overline{b},c,\overline{c},\overline{c})$
	$(\overline{a},b,\overline{c},c,\overline{c},a,\overline{c},\overline{a},\overline{b},\overline{b})$	$(\overline{a},\overline{b},\overline{c},c,b,\overline{c},b,c,\overline{c},\overline{b})$
$(7,11,21)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{a},\overline{b},\overline{b},b,\overline{c},c,\overline{c},\overline{c},c,\overline{a})$
	$(\overline{a},\overline{c},b,c,\overline{a},\overline{b},\overline{b},c,\overline{c},\overline{b})$	$(\overline{a},\overline{a},b,\overline{b},c,\overline{c},\overline{a},b,c,b)$
$(7,12,20)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{a},\overline{c},c,c,\overline{c},a,\overline{a},\overline{b},\overline{b},\overline{b})$
	$(\overline{a},c,\overline{b},\overline{c},b,b,c,\overline{c},b,\overline{b})$	$(\overline{a},\overline{a},\overline{b},b,\overline{b},\overline{a},b,c,\overline{c},c)$
$(7,16,16)$	$(0,\overline{b},\overline{b},\overline{c},\overline{b},b,\overline{b},\overline{c},\overline{b},\overline{b})$	$(\overline{a},\overline{c},b,\overline{c},\overline{c},a,c,\overline{a},\overline{b},\overline{b})$
	$(\overline{a},c,\overline{c},c,\overline{b},\overline{c},\overline{c},\overline{b},b,\overline{c})$	$(\overline{a},\overline{c},\overline{a},c,\overline{b},\overline{a},b,\overline{c},\overline{b},c)$
$(8,8,23)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{b},\overline{c},\overline{c},\overline{a},c,\overline{c},a,c,\overline{c},\overline{b})$
	$(\overline{b},\overline{b},c,\overline{b},\overline{c},c,a,a,\overline{a},c)$	$(\overline{b},\overline{a},\overline{c},c,\overline{c},b,a,\overline{b},a,\overline{c})$
$(8,9,22)$	$(0,\overline{c},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{c})$	$(\overline{a},\overline{b},\overline{c},\overline{c},c,b,\overline{c},\overline{a},c,\overline{c})$
	$(\overline{a},b,\overline{b},\overline{a},\overline{b},\overline{b},c,\overline{c},\overline{a},\overline{c})$	$(\overline{a},b,\overline{a},\overline{c},c,b,\overline{a},\overline{b},c,\overline{c})$
$(8,13,18)$	$(0,\overline{c},\overline{c},\overline{b},\overline{c},c,\overline{c},\overline{b},\overline{c},\overline{c})$	$(\overline{a},\overline{b},c,\overline{b},\overline{c},\overline{c},\overline{a},\overline{c},b,\overline{b})$
	$(\overline{a},\overline{b},\overline{a},c,b,\overline{c},c,c,\overline{c},\overline{b})$	$(\overline{a},\overline{a},b,b,\overline{c},b,\overline{c},a,\overline{a},b)$
$(9,15,15)$	$(0,\overline{b},\overline{b},\overline{c},\overline{b},b,\overline{b},\overline{c},\overline{b},\overline{b})$	$(\overline{c},b,\overline{c},b,\overline{c},\overline{b},b,b,\overline{b},\overline{b})$
	$(\overline{a},\overline{c},\overline{a},c,\overline{a},c,a,b,\overline{a},c)$	$(\overline{a},\overline{a},\overline{c},\overline{a},a,\overline{c},c,c,\overline{c},\overline{c})$
$(10,14,15)$	$(0,\overline{c},\overline{c},\overline{b},\overline{c},b,\overline{c},\overline{b},\overline{c},\overline{c})$	$(\overline{a},\overline{a},\overline{b},\overline{a},\overline{c},c,a,\overline{c},\overline{b},b)$
	$(\overline{a},b,\overline{c},b,\overline{a},b,c,\overline{c},\overline{b},\overline{a})$	$(\overline{a},\overline{b},c,\overline{b},a,\overline{c},\overline{a},c,\overline{b},\overline{b})$
$(11,11,17)$	$(0,\overline{c},\overline{c},\overline{b},\overline{c},c,\overline{c},\overline{b},\overline{c},\overline{c})$	$(\overline{b},\overline{b},\overline{a},\overline{a},a,c,c,\overline{b},c,\overline{b})$
	$(\overline{b},\overline{a},a,\overline{c},\overline{c},c,a,a,\overline{b},a)$	$(\overline{b},a,\overline{b},\overline{a},c,\overline{b},\overline{c},a,c,\overline{c})$
$(11,12,16)$	$(0,\overline{c},\overline{c},\overline{a},c,\overline{c},c,\overline{a},\overline{c},\overline{c})$	$(\overline{a},\overline{b},\overline{a},\overline{b},\overline{c},\overline{c},a,\overline{c},\overline{a},b)$
	$(\overline{a},\overline{c},\overline{b},b,\overline{c},c,b,\overline{a},\overline{b},c)$	$(\overline{a},b,\overline{a},c,\overline{b},\overline{a},b,c,\overline{b},\overline{b})$
$(12,12,15)$	$(0,\overline{b},\overline{c},\overline{c},\overline{c},c,\overline{c},\overline{c},\overline{c},\overline{b})$	$(\overline{b},\overline{c},c,\overline{b},\overline{a},\overline{a},\overline{a},\overline{c},c,\overline{b})$
	$(\overline{b},a,\overline{b},a,\overline{b},c,\overline{a},\overline{b},\overline{a},a)$	$(\overline{b},\overline{b},a,\overline{a},\overline{c},a,c,\overline{a},\overline{b},\overline{c})$
$(12,13,14)$	$(0,\overline{c},\overline{a},\overline{c},a,\overline{c},a,\overline{c},\overline{a},\overline{c})$	$(\overline{b},\overline{b},\overline{c},c,c,\overline{c},\overline{c},b,\overline{b},\overline{b})$
	$(\overline{a},\overline{b},\overline{c},\overline{c},b,\overline{a},b,c,\overline{c},b)$	$(\overline{a},\overline{a},\overline{a},\overline{b},b,\overline{a},b,b,a,\overline{a})$

References

[1] G. Berman, Families of skew-circulant weighing matrices, *Ars Combinatoria* **4** (1977), 293–307.

[2] A. V. Geramita and Jennifer Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York-Basel, 1979.

[3] W. H. Holzmann and H. Kharaghani, On the orthogonal designs of order 24, *Discrete Appl. Math.* **102** (2000), 103–114.

[4] W. H. Holzmann and H. Kharaghani, On the orthogonal designs of order 40, *J. Statist. Plann. Inference* **96** (2001), 415–429.

[5] W. H. Holzmann and H. Kharaghani, On the Plotkin arrays, *Australas. J. Combin.* **22** (2000), 287–299.

[6] W. H. Holzmann and H. Kharaghani, J. Seberry, and B. Tayfeh-Rezaie, On orthogonal designs in order 48, *J. Statist. Plann. Inference*, to appear.

[7] D. Jungnickel, On automorphism groups of divisible designs, *Canadian J. Math.* **34** (1982), 257–297.

[8] D. Jungnickel and V. Tonchev, Perfect codes and balanced weighing matrices, *Finite Fields and their Applications* **5** (1999), 294–300.

[9] H. Kharaghani, Arrays for orthogonal designs, *J. Combin. Des.* **8** (2000), 127–130.