On Orthogonal Designs in Order 48

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Abstract

We show that all 3164 possible $OD(48; s_1, s_2, s_3)$ exist. In addition to the use of some classical techniques we employ two new methods of construction.

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1 Introduction

An *orthogonal design* A, of order n, and type $(s_1, s_2, \dots, s_u)$, denoted $OD(n; s_1, s_2, \dots, s_u)$ on the commuting variables $\pm x_1, \pm x_2, \dots, \pm x_u$ is a square matrix of order n with entries $\pm x_k$ or 0, where each x_k occurs s_k times in each row and column such that the distinct rows are pairwise orthogonal. In other words

$$AA^T = (s_1 x_1^2 + \cdots + s_u x_u^2) I_n,$$

where I_n is the identity matrix. It is known that the maximum number of variables in an orthogonal design is $\rho(n)$, the Radon number, where for $n = 2^a b$, b odd, set $a = 4c + d, 0 \leq d < 4$, then $\rho(n) = 8c + 2^d$.

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It is conjectured that all possible 3-tuples are types of an orthogonal design of order $8n$. This conjecture has been verified for $n = 1, 2, 3, 4$. In this paper we will show that the conjecture also holds for $n = 6$. In addition to the use of some classical techniques we employ two new methods of construction.

Section 2 is devoted to classical results. By classical results we mean all orthogonal designs obtained more than 25 years ago. Out of 3164 possible cases 3147 are constructed using classical results. To show the evolution of the subject area, we keep the new results out of this section. By new results we mean results obtained in the past four years.

2 Classical results

For this part we make extensive use of the book of Geramita and Seberry [2]. For convenience we quote a number of construction methods from this book.

Theorem 1 (Equating and Killing Theorem) *If A is an orthogonal design $OD(n; s_1, s_2, \ldots, s_u)$ on the commuting variables $\pm x_1, \pm x_2, \ldots, \pm x_u$, then there is an orthogonal design $OD(n; s_1, s_2, \ldots, s_i + s_j, \ldots, s_u)$ and $OD(n; s_1, s_2, \ldots, s_{j-1}, s_{j+1}, \ldots, s_u)$ on the $u - 1$ commuting variables $\pm x_1, \pm x_2, \ldots, \pm x_{j-1}, \pm x_{j+1}, \ldots, \pm x_u$.*

Theorem 2 (Multiplication Theorem) *If there exists an orthogonal design $OD(n; s_1, s_2, \ldots, s_u)$, then there exists an orthogonal design $OD(2n; s_1, s_1, es_2, \ldots, es_u)$, where $e = 1$ or 2.*

Theorem 3 (Second Multiplication Theorem) *If there exists an orthogonal design $OD(n; s_1, s_2, \ldots, s_u)$, then there exists an orthogonal design $OD(2n; e_1 s_1, e_2 s_2, \ldots, e_u s_u)$, where $e_i = 1$ or 2, $i = 1, \ldots, u$.*

Theorem 4 *We use the following designs in order 24 and the Multiplication Theorems above, to obtain the designs given in the Appendix which have 9, 8, and 7 variables in order 48.*

8-Variables	Label	7-Variables	Label	6-Variables	Label
1 1 1 1 1 1 1 1	I	1 1 1 1 1 4 5	IX	1 1 1 2 4 10	XI
1 1 1 1 1 1 1 4	II	1 1 1 1 2 2 8	X	1 1 1 3 4 9	XII
1 1 1 1 1 1 9 9	III			1 1 1 4 6 6	XIII
1 1 1 1 1 2 8 9	IV			1 1 2 3 4 9	XIV
1 1 1 1 1 4 4 4	V				
1 1 1 1 1 5 5 9	VI				
1 1 1 1 4 4 4 4	VII				
3 3 3 3 3 3 3 3	VIII				

Table 1 Orthogonal designs in order 24

Theorem 5 *An $OD(6; a, b)$ and an $AOD(8; (a_1, a_2, \ldots, a_s); (b_1, b_2, \ldots, b_t))$ give an $OD(48; aa_1, aa_2, \ldots, aa_s, bb_1, bb_2, \ldots, bb_t)$.*

Table 2 gives the designs in order 48 constructed using Theorem 5.

New Design	$OD(6; a, b)$	$AOD(8; (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4))$
(1 1 2 2 2 4 4)	(1, 1)	(1, 1, 2, 4);(2, 2, 4)
(1 1 2 2 2 32)	(1, 4)	(1, 1, 2, 2, 2);(8)
(1 1 2 4 8 8 16)	(1, 4)	(1, 1, 2, 4);(2, 2, 4)
(1 1 7 7)	(1, 1)	(1, 7);(1, 7)
(1 2 2 2 3 6)	(1, 1)	(1, 2, 2, 3);(2, 6)
(1 2 2 3 8 24)	(1, 4)	(1, 2, 2, 3);(2, 6)
(1 4 7 28)	(1, 4)	(1, 7);(1, 7)
(2 2 4 4 4 8 16)	(4, 1)	(1, 1, 2, 4);(2, 2, 4)
(2 4 4 4 6 12)	(2, 2)	(1, 2, 2, 3);(2, 6)
(2 4 6 8 8 12)	(4, 1)	(1, 2, 2, 3);(2, 6)
(4 4 8 8 8 8)	(4, 1)	(1, 1, 2, 2, 2);(8)

Table 2 Designs in order 48 made using Theorem 5

Theorem 6 *An* $OD(12; a, b, c, d)$ *and an* $AOD(4; (a_1, a_2, a_3, a_4); (\beta))$ *give an* $OD(48; aa_1, aa_2, aa_3, aa_4, \beta b, \beta c, \beta d)$.

Remark 1 This theorem does not give any designs in order 48 not able to be constructed by other methods.

Theorem 7 *An* $OD(12; a, b)$ *and an* $AOD(4; (a_1, a_2, \ldots, a_s); (b_1, b_2, \ldots, b_t))$ *give an* $OD(48; aa_1, aa_2, \ldots, aa_s, bb_1, bb_2, \ldots, bb_t)$.

Corollary 1 *Since every orthogonal design* $OD(12; a, b)$ *exists* $1 \leq a,\ b,\ a + b \leq 12$ *except* $(a, b) = (1, 7)$, $(3, 5)$ *and* $(4, 7)$ *and since* $AOD(4; (1, 1, 1, 1), (1))$, $AOD(4; (1), (1, 1, 1, 1))$, $AOD(4; (1, 1, 1), (1, 1, 1))$, *and* $AOD(4; (1, 1, 2), (1, 1, 2))$ *exist we have the existence of all* $OD(48; a, a, a, a, b)$, $OD(48; a, a, a, b, b, b)$ *and* $OD(48; a, a, 2a, b, b, 2b)$ *for the given* a *and* b.

Theorem 8 (Robinson [2, p267]) *An* $AOD(4; (u_1, u_2, \ldots, u_s); (v, w_1, \ldots, w_t))$ *and a product design of type* $PD(48; (1, 1, 1; 1, 1, 1; 9))$ *give one of*

(*i*) $OD(48; v, v, v, w, w, w, 9u_1, 9u_2, \ldots, 9u_s)$;

(*ii*) $OD(48; v, v, v, bw_1, bw_2, \ldots, bw_t, 9u_1, 9u_2, \ldots, 9u_s)$, $b = 1,\ 2$ *or* 3.

Theorem 9 *If there are* $OD(n; a_1, a_2, \ldots, a_s)$ *and suitable* 3×3 *matrices, then there are orthogonal designs on* s *or fewer variables in order* $3n$.

Table 3 gives the designs in order 16 that we use to construct designs by the use of Theorem 9.

9-Variables	Label	8-Variables	Label
1 1 1 1 1 1 1 1 1	16a	1 1 1 1 1 1 3 3	16j
1 1 1 1 1 1 1 1 2	16b	1 1 1 1 1 1 4 4	16k
1 1 1 1 1 1 1 1 4	16c	1 1 1 1 1 1 5 5	16ℓ
1 1 1 1 1 1 1 1 8	16d	1 1 1 1 2 2 3 3	16m
1 1 1 1 1 1 2 2 2	16e	1 1 1 1 2 2 4 4	16n
1 1 1 1 1 2 2 2 2	16f	1 1 2 2 2 2 3 3	16o
1 1 1 1 1 2 3 3 3	16g		
1 1 1 1 2 2 2 2 2	16h		
1 1 2 2 2 2 2 2 2	16i		
8-Variables Derived by Thm1	Label	8-Variables Derived by Thm1	Label
1 1 1 1 1 1 1 1	16a1	1 1 1 1 1 3 3 5	16g3
1 1 1 1 1 1 1 2	16a2	1 1 1 1 2 2 2 2	16e4
1 1 1 1 1 1 1 3	16b1	1 1 1 1 2 2 2 3	16f2
1 1 1 1 1 1 1 4	16c1	1 1 1 1 2 2 2 4	16h1
1 1 1 1 1 1 1 5	16c2	1 1 1 1 2 3 3 3	16g4
1 1 1 1 1 1 1 8	16d1	1 1 1 1 2 3 3 4	16g5
1 1 1 1 1 1 1 9	16d2	1 1 1 1 3 3 3 3	16g6
1 1 1 1 1 1 2 2	16b2	1 1 1 2 2 2 2 2	16f4
1 1 1 1 1 1 2 4	16c3	1 1 1 2 2 2 2 3	16h2
1 1 1 1 1 1 2 8	16d3	1 1 1 2 2 3 3 3	16g7
1 1 1 1 1 2 2 2	16e1	1 1 2 2 2 2 2 2	16i1
1 1 1 1 1 2 2 3	16e2	1 1 2 2 2 2 2 4	16i2
1 1 1 1 1 2 2 4	16f3	1 2 2 2 2 2 2 2	16i3
1 1 1 1 1 2 3 3	16g1	1 2 2 2 2 2 2 3	16i4
1 1 1 1 1 2 3 6	16g2	2 2 2 2 2 2 2 2	16i5
1 1 1 1 1 3 3 3	16g3		

Table 3 Designs in order 16 used by Theorem 9

Theorem 10 *If A is an $OD(24; a_1, a_2, \ldots, a_s)$, then*

$$D = \begin{pmatrix} xI & A \\ -A^t & xI \end{pmatrix},$$

where A^t means the transpose of A, is an $OD(48; 1, a_1, a_2, \ldots, a_s)$.

Theorem 11 *If A is a symmetric $OD(n; a_1, a_2, \ldots, a_s)$ and $B, C = 0$ or A, then*

$$D = \begin{pmatrix} xI & A & B & C \\ -A & xI & C & -B \\ -B & -C & xI & A \\ -C & B & -A & xI \end{pmatrix},$$

gives $OD(4n; 1, a_1, a_2, \ldots, a_s)$, $OD(4n; 1, 2a_1, 2a_2, \ldots, 2a_s)$, and $OD(4n; 1, 3a_1, 3a_3, \ldots, 3a_s)$.

Theorem 12 *If $A = OD(n; a_1, a_2, \ldots, a_s)$ is an orthogonal design, then replacing the variable of an $OD(6; 5)$ by A gives an $OD(6n; 5a_1, 5a_2, \ldots, 5a_s)$. If A is symmetric, then*

$$D = \begin{pmatrix} 0 & A & A & xI & A & -A \\ A & 0 & A & A & -A & xI \\ A & A & 0 & -A & xI & A \\ & & & & & \\ -xI & -A & A & 0 & A & A \\ -A & A & -xI & A & 0 & A \\ A & -xI & -A & A & A & 0 \end{pmatrix}$$

and if A is skew symmetric, then

$$D = \begin{pmatrix} xI & A & A & 0 & A & -A \\ A & xI & A & A & -A & 0 \\ A & A & xI & -A & 0 & A \\ & & & & & \\ 0 & A & -A & xI & -A & -A \\ A & -A & 0 & -A & xI & -A \\ -A & 0 & A & -A & -A & xI \end{pmatrix}$$

give an $OD(6n; 1, 4a_1, 4a_2, \ldots, 4a_s)$.

Corollary 2 *The $OD(8; 1, 1, 1, 1, 1, 1, 1, 1)$ gives an $OD(48; 5, 5, 5, 5, 5, 5, 5, 5)$. The $OD(8; 1, 1, 1, 1, 1, 1, 1)$ may be made skew symmetric and used to form the $OD(48; 1, 4, 4, 4, 4, 4, 4, 4)$.*

Theorem 13 *All $OD(24; s_1, s_2, 24-s_1-s_2)$ exist so all $OD(48; t_1, t_2, 48-t_1-t_2)$ exist.*

Some of designs of order 48 in 5 to 9 variables obtained from Theorems 2–12 are listed in the Appendix. Using them and Theorems 1 and 13, we have the following theorem.

Theorem 14 *All $OD(48; s_1, s_2, s_3)$, $0 \le s_1, s_2, s_3, s_1 + s_2 + s_3 \le 48$, can be constructed using classical results except possibly the following 17 types:*

1	5	41	5	17	25	7	11	29	7	17	23	15	15	15.
5	5	37	7	7	27	7	13	21	9	17	17			
5	11	31	7	7	33	7	13	25	11	17	17			
5	13	23	7	11	23	7	15	25	13	17	17			

3 Construction using 16 circulant matrices

Following [7], a set $\{A_1, A_2, \ldots, A_{2n}\}$ of square real matrices is said to be amicable if

$$\sum_{i=1}^{n} (A_{\sigma(2i-1)} A^t_{\sigma(2i)} - A_{\sigma(2i)} A^t_{\sigma(2i-1)}) = 0,$$

for some permutation σ of the set $\{1, 2, \dots, 2n\}$.

For simplicity, we will always take $\sigma(i) = i$ unless otherwise specified. Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general.

Theorem 15 [7, Theorem 4 and Example 6] *Let $\{A_i\}_{i=1}^{8}$ be an amicable set of circulant matrices such that $\sum_{i=1}^{4}(A_{2i-1}A_{2i-1}^t + 3A_{2i}A_{2i}^t)$ is a multiple of the identity matrix. Then the matrix*

$$H = \left(\begin{array}{cccccccccccccccc}
A_1 & A_2 & A_2 & A_2 & -A_3R & A_4R & A_4R & A_4R & -A_5R & A_6R & A_6R & A_6R & -A_7R & A_8R & A_8R & A_8R \\
-A_2 & A_1 & -A_2 & A_2 & A_4R & A_4R & -A_4R & A_3R & A_6R & A_6R & -A_6R & A_5R & A_8R & A_8R & -A_8R & A_7R \\
-A_2 & A_2 & A_1 & -A_2 & A_4R & -A_4R & A_3R & A_4R & A_6R & -A_6R & A_5R & A_6R & A_8R & -A_8R & A_7R & A_8R \\
-A_2 & -A_2 & A_2 & A_1 & A_4R & A_3R & A_4R & -A_4R & A_6R & A_5R & A_6R & -A_6R & A_8R & A_7R & A_8R & -A_8R \\
A_3R & -A_4R & -A_4R & -A_4R & A_1 & A_2 & A_2 & A_2 & -{A_7}^tR & -{A_8}^tR & -{A_8}^tR & -{A_8}^tR & {A_5}^tR & {A_6}^tR & {A_6}^tR & {A_6}^tR \\
-A_4R & -A_4R & A_4R & -A_3R & -A_2 & A_1 & -A_2 & A_2 & -{A_8}^tR & -{A_8}^tR & {A_8}^tR & {A_7}^tR & {A_6}^tR & {A_6}^tR & -{A_6}^tR & -{A_5}^tR \\
-A_4R & A_4R & -A_3R & -A_4R & -A_2 & A_2 & A_1 & -A_2 & -{A_8}^tR & {A_8}^tR & {A_7}^tR & -{A_8}^tR & {A_6}^tR & -{A_6}^tR & -{A_5}^tR & {A_6}^tR \\
-A_4R & -A_3R & -A_4R & A_4R & -A_2 & -A_2 & A_2 & A_1 & -{A_8}^tR & {A_7}^tR & -{A_8}^tR & {A_8}^tR & {A_6}^tR & -{A_5}^tR & {A_6}^tR & -{A_6}^tR \\
A_5R & -A_6R & -A_6R & -A_6R & {A_7}^tR & {A_8}^tR & {A_8}^tR & {A_8}^tR & A_1 & A_2 & A_2 & A_2 & -{A_3}^tR & -{A_4}^tR & -{A_4}^tR & -{A_4}^tR \\
-A_6R & -A_6R & A_6R & -A_5R & {A_8}^tR & {A_8}^tR & -{A_8}^tR & -{A_7}^tR & -A_2 & A_1 & -A_2 & A_2 & -{A_4}^tR & -{A_4}^tR & {A_4}^tR & {A_3}^tR \\
-A_6R & A_6R & -A_5R & -A_6R & {A_8}^tR & -{A_8}^tR & -{A_7}^tR & {A_8}^tR & -A_2 & A_2 & A_1 & -A_2 & -{A_4}^tR & {A_4}^tR & {A_3}^tR & -{A_4}^tR \\
-A_6R & -A_5R & -A_6R & A_6R & {A_8}^tR & -{A_7}^tR & {A_8}^tR & -{A_8}^tR & -A_2 & -A_2 & A_2 & A_1 & -{A_4}^tR & {A_3}^tR & -{A_4}^tR & {A_4}^tR \\
A_7R & -A_8R & -A_8R & -A_8R & -{A_5}^tR & -{A_6}^tR & -{A_6}^tR & -{A_6}^tR & {A_3}^tR & {A_4}^tR & {A_4}^tR & {A_4}^tR & A_1 & A_2 & A_2 & A_2 \\
-A_8R & -A_8R & A_8R & -A_7R & -{A_6}^tR & -{A_6}^tR & {A_6}^tR & {A_5}^tR & {A_4}^tR & {A_4}^tR & -{A_4}^tR & -{A_3}^tR & -A_2 & A_1 & -A_2 & A_2 \\
-A_8R & A_8R & -A_7R & -A_8R & -{A_6}^tR & {A_6}^tR & {A_5}^tR & -{A_6}^tR & {A_4}^tR & -{A_4}^tR & -{A_3}^tR & {A_4}^tR & -A_2 & A_2 & A_1 & -A_2 \\
-A_8R & -A_7R & -A_8R & A_8R & -{A_6}^tR & {A_5}^tR & -{A_6}^tR & {A_6}^tR & {A_4}^tR & -{A_3}^tR & {A_4}^tR & -{A_4}^tR & -A_2 & -A_2 & A_2 & A_1
\end{array}\right),$$

is an orthogonal matrix, where R is the back–diagonal identity matrix of an appropriate order.

We now apply an algorithm developed in [3, 4, 5] which uses a known set of circulant matrices of order 3 of type (s_1, s_2, s_3, s_4) to construct an amicable set of eight matrices suitable for the array H in Theorem 15.

Start with a set of four circulant matrices A, B, C, and D in variables a, b, c, and d for which $AA^t + BB^t + CC^t + DD^t = (s_1a^2 + s_2b^2 + s_3c^2 + s_4d^2)I_3$. From matrices A, B, C, and D construct four new circulant matrices E, F, G, and H by replacing a by e, b by f, c by g, and d by h. The new matrices obviously satisfy an additive property as above but in variables e, f, g, and h. We then search to find a pairing, while reducing the number of variables to four by collapsing different variables, between the sets $\{A, B, C, D\}$ and $\{E, F, G, H\}$ to form an amicable set of eight matrices. All but two of the amicable sets listed in Table 5 were obtained from either the second or the third of the sequences listed in Table 4.

The amicable sets for $(1, 7, 7, 33)$ and $(1, 7, 17, 23)$ required a refinement of the above approach. Instead of constructing E, F, G, and H from A, B, C, and D, we selected E, F, G, and H to be another set of four circulant matrices obtained from those in Table 4 and in addition allowed for circular permutation of the entries. In the case of $(1, 7, 7, 33)$ a matching was found between the matrices obtained from the $(1, 1, 1, 9)$ line in Table 4 and a permute of $(2, 2, 4, 4)$ while for $(1, 7, 17, 23)$ a matching was found between the matrices for $(1, 1, 5, 5)$ and $(1, 2, 3, 6)$.

To obtain amicable sets for $(5, 13, 23)$ and $(7, 13, 21)$ as listed in Table 6, the above approach was extended by considering known non-full orthogonal designs. Specifically, these two sets were obtained from the third, respectively the first, of the sets listed in Table 4 after zeroing out one of the variables, matched with the set $(a, c, -c), (b, d, -d), (c, c, 0), (d, d, 0)$ of type $(1, 1, 4, 4)$. Other combinations did not generate any new types.

Amicable sets of the above type, where A, B, C, and D are matched with E, F, G, and H are called special amicable sets (see [4]).

Theorem 16 *All of the* 17 *orthogonal designs of order* 48 *undecided in the Theorem* 14*, except possibly those of the following types, are constructible from the array H above:*

7	7	27	9	17	17	13	17	17.
7	11	23	11	17	17			

Remark 2 As noted in [4, Theorem 3], special amicable sets can be used in all arrays obtained by Kharaghani in [7]. Consequently each gives infinitely many new orthogonal designs.

type	A	B	C	D
$(1,1,1,9)$	$(a,d,\overline{d})$	$(b,d,\overline{d})$	$(c,d,\overline{d})$	(d,d,d)
$(1,1,2,8)$	$(a,d,\overline{d})$	$(b,d,\overline{d})$	(c,d,d)	$(\overline{c},d,d)$
$(1,1,5,5)$	$(a,c,\overline{c})$	$(b,d,\overline{d})$	(c,d,d)	$(\overline{d},c,c)$
$(1,2,3,6)$	$(a,d,\overline{d})$	(c,d,d)	$(c,\overline{d},b)$	$(c,\overline{b},\overline{d})$
$(2,2,4,4)$	(a,c,d)	$(a,\overline{d},\overline{c})$	$(b,c,\overline{d})$	$(b,\overline{c},d)$
$(3,3,3,3)$	(a,b,c)	$(\overline{b},a,d)$	$(\overline{c},\overline{d},a)$	$(\overline{d},c,\overline{b})$

Table 4 Known [1, p348] full OD of order 12 in 4 variables

type	A_1 A_2	A_3 A_4	A_5 A_6	A_7 A_8
$(1,1,5,41)$	$(a,d,\overline{d})$	$(b,d,\overline{d})$	(c,d,d)	$(\overline{c},d,d)$
	(d,d,d)	$(\overline{d},d,d)$	$(c,d,\overline{d})$	$(d,d,\overline{d})$
$(1,5,5,37)$	$(a,c,\overline{c})$	$(d,b,\overline{b})$	(c,b,b)	$(\overline{b},c,c)$
	(d,d,d)	$(\overline{d},d,d)$	$(d,d,\overline{d})$	$(d,d,\overline{d})$
$(1,5,11,31)$	$(a,c,\overline{c})$	$(d,c,\overline{c})$	(b,c,c)	$(\overline{b},c,c)$
	(d,d,d)	$(\overline{d},d,d)$	$(b,d,\overline{d})$	$(c,d,\overline{d})$
$(1,5,17,25)$	$(a,c,\overline{c})$	$(d,c,\overline{c})$	(b,c,c)	$(\overline{b},c,c)$
	(c,d,d)	$(\overline{c},d,d)$	$(b,d,\overline{d})$	$(c,d,\overline{d})$
$(1,7,7,33)$	$(\overline{b},\overline{d},d)$	$(a,d,\overline{d})$	$(c,d,\overline{d})$	(d,d,d)
	$(\overline{d},c,d)$	(b,d,d)	$(\overline{d},b,\overline{d})$	$(c,\overline{d},d)$
$(1,7,11,29)$	$(b,c,\overline{c})$	$(a,c,\overline{c})$	(d,c,c)	$(\overline{d},c,c)$
	(b,d,d)	$(\overline{b},d,d)$	$(d,d,\overline{d})$	$(c,d,\overline{d})$
$(3,7,13,25)$	(c,c,c)	$(\overline{c},c,c)$	$(b,c,\overline{c})$	$(d,c,\overline{c})$
	$(a,d,\overline{d})$	$(c,d,\overline{d})$	(b,d,d)	$(\overline{b},d,d)$
$(1,7,15,25)$	$(a,d,\overline{d})$	$(b,d,\overline{d})$	(d,d,d)	$(\overline{d},d,d)$
	(d,c,c)	$(\overline{c},d,d)$	$(b,d,\overline{d})$	$(b,c,\overline{c})$
$(1,7,17,23)$	$(a,c,\overline{c})$	$(b,d,\overline{d})$	(c,d,d)	$(\overline{d},c,c)$
	(c,d,d)	$(c,d,\overline{d})$	$(\overline{d},c,\overline{b})$	$(\overline{d},b,c)$
$(3,15,15,15)$	(b,b,b)	$(\overline{b},b,b)$	$(b,b,\overline{b})$	$(b,b,\overline{b})$
	$(b,c,\overline{c})$	$(a,d,\overline{d})$	(c,d,d)	$(\overline{d},c,c)$

Table 5 Amicable sets for some full OD of order 48 in 4 variables

type	A_1 A_2	A_3 A_4	A_5 A_6	A_7 A_8
$(5,13,23)$	$(b,a,\overline{a})$	$(a,a,\overline{c})$	(a,c,c)	$(c,\overline{c},0)$
	$(0,c,c)$	$(\overline{b},\overline{b},0)$	$(\overline{c},\overline{c},c)$	$(b,\overline{b},c)$
$(7,13,21)$	$(a,c,\overline{c})$	$(b,c,\overline{c})$	(c,c,c)	$(c,\overline{c},0)$
	$(0,b,b)$	$(\overline{a},\overline{c},c)$	$(\overline{c},\overline{c},0)$	$(b,\overline{b},a)$

Table 6 Amicable sets for some OD of order 48 in 3 variables

4 Construction using 4 negacyclic matrices

Having exhausted all possible avenues and still having five orthogonal designs unresolved forced us to look to alternatives. Obviously order 48 is too large for a non-structured search. Our previous experience, see [6], had proven that negacyclic matrices are much more versatile for searches in the case where the order of block matrices is even. A negacyclic matrix of order n is any polynomial in the negashift matrix U of order n, where $U = [u_{ij}]$, $u_{i(i+1)} = 1$, $i = 1, 2, \cdots, n-1$, $u_{n1} = -1$, and zero otherwise. We searched for four negacyclic matrices A_i, $i = 1, 2, 3, 4$ of order 12 in three variables satisfying a suitable additive property. As is known, we then substituted them in the Goethals-Seidel array

$$G = \begin{pmatrix} A_1 & A_2R & A_3R & A_4R \\ -A_2R & A_1 & A_4^tR & -A_3^tR \\ -A_3R & -A_4^tR & A_1 & A_2^tR \\ -A_4R & A_3^tR & -A_2^tR & A_1 \end{pmatrix},$$

to get the remaining orthogonal designs. Table 7 shows the first row of each of A_i, $i = 1, 2, 3, 4$. Note that the two orthogonal designs of types $(5, 13, 23)$ and $(7, 13, 21)$ were also obtained using four negacyclic matrices of order 12.

Corollary 3 *All* 3164 *possible orthogonal designs of order* 48 *in three variables exist.*

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type	A_1 A_2 A_3 A_4
$(5,13,23)$	$(\bar{c},\bar{c},\bar{c},\bar{c},\bar{c},\bar{c},\bar{c},c,c,\bar{c},\bar{c},\bar{c})$ $(\bar{c},\bar{c},c,\bar{c},c,\bar{c},c,\bar{c},\bar{c},c,0,\bar{c})$ $(\bar{a},\bar{b},\bar{a},b,b,b,0,\bar{b},b,b,\bar{a},b)$ $(\bar{a},0,\bar{b},0,\bar{a},\bar{b},b,0,0,0,b,\bar{b})$
$(7,11,23)$	$(\bar{c},\bar{c},\bar{c},\bar{c},\bar{c},\bar{c},\bar{c},c,c,\bar{c},\bar{c},\bar{c})$ $(\bar{b},\bar{c},c,c,\bar{c},c,\bar{c},c,\bar{c},c,c,\bar{c})$ $(\bar{a},\bar{b},\bar{a},b,\bar{a},b,0,\bar{b},a,\bar{b},\bar{a},\bar{b})$ $(\bar{a},\bar{b},0,\bar{b},0,0,\bar{a},\bar{b},0,b,0,0)$
$(7,13,21)$	$(\bar{c},\bar{c},\bar{c},\bar{c},\bar{c},\bar{c},c,\bar{c},c,c,\bar{c},\bar{c})$ $(\bar{a},\bar{c},\bar{a},0,0,\bar{b},\bar{b},\bar{c},c,\bar{b},\bar{a},0)$ $(\bar{a},\bar{b},0,0,b,c,\bar{a},0,\bar{b},b,c,\bar{c})$ $(\bar{a},b,0,b,\bar{a},c,\bar{b},\bar{b},\bar{c},b,\bar{b},c)$
$(7,7,27)$	$(\bar{c},\bar{c},\bar{c},\bar{c},\bar{c},\bar{c},\bar{c},c,c,\bar{c},\bar{c},\bar{c})$ $(\bar{a},\bar{a},\bar{c},\bar{b},\bar{c},c,\bar{c},0,b,\bar{c},b,\bar{a})$ $(\bar{a},c,\bar{a},0,b,b,\bar{b},\bar{a},\bar{c},c,0,0)$ $(\bar{a},\bar{c},c,\bar{b},\bar{c},0,c,\bar{c},c,0,0,\bar{c})$
$(9,17,17)$	$(\bar{b},\bar{c},\bar{c},\bar{c},\bar{c},c,c,\bar{c},\bar{c},c,\bar{c},c)$ $(\bar{b},\bar{b},\bar{c},\bar{b},0,c,\bar{c},\bar{c},b,\bar{c},0,\bar{b})$ $(\bar{b},\bar{b},\bar{b},\bar{b},c,\bar{b},b,\bar{b},b,b,\bar{b},\bar{b})$ $(\bar{a},\bar{a},\bar{a},0,a,\bar{a},0,\bar{a},\bar{a},0,a,\bar{a})$
$(11,17,17)$	$(\bar{b},\bar{c},\bar{c},\bar{c},\bar{c},c,c,\bar{c},\bar{c},c,\bar{c},c)$ $(\bar{b},\bar{b},\bar{c},\bar{b},0,c,\bar{c},\bar{c},b,\bar{c},0,\bar{b})$ $(\bar{b},\bar{b},\bar{b},\bar{b},c,\bar{b},b,\bar{b},b,b,\bar{b},\bar{b})$ $(\bar{a},\bar{a},\bar{a},\bar{a},0,\bar{a},a,\bar{a},a,a,\bar{a},\bar{a})$
$(13,17,17)$	$(0,\bar{a},\bar{a},\bar{b},\bar{c},\bar{a},a,\bar{a},\bar{c},\bar{b},\bar{a},\bar{a})$ $(\bar{a},b,\bar{a},c,b,\bar{b},\bar{a},c,b,b,\bar{a},\bar{c})$ $(\bar{a},c,\bar{b},\bar{b},c,\bar{c},\bar{b},c,c,\bar{b},\bar{b},\bar{c})$ $(\bar{a},\bar{c},\bar{b},b,\bar{c},c,\bar{b},\bar{c},\bar{c},b,\bar{b},c)$

Table 7 First rows of four negacyclic matrices of order 12 in 3 variables

5 Appendix

9-Variables	Ref	9-Variables	Ref
1 1 1 1 1 1 1 1 1	Thm2,I	1 1 1 1 2 3 3 3 9	Thm9
1 1 1 1 1 1 1 1 2	Thm9,16b	1 1 1 1 4 4 4 4 4	Thm2,VII
1 1 1 1 1 1 1 1 4	Thm2,II	1 1 1 2 2 2 2 2 4	Thm9,16h
1 1 1 1 1 1 1 1 8	Thm9,16d	1 1 1 2 2 2 2 2 8	Thm9,16f
1 1 1 1 1 1 1 2 4	Thm9,16b	1 1 1 2 2 2 9 9 9	Thm8
1 1 1 1 1 1 1 4 4	Thm2,II	1 1 1 2 2 2 9 9 18	Thm8
1 1 1 1 1 1 1 4 8	Thm9,16d	1 1 1 2 2 3 3 3 8	Thm9,16g
1 1 1 1 1 1 1 9 9	Thm2,III	1 1 1 3 3 3 9 9 9	Thm8
1 1 1 1 1 1 2 2 2	Thm9,16e	1 1 1 3 3 3 9 9 18	Thm8
1 1 1 1 1 1 2 2 8	Thm9,16e	1 1 2 2 2 2 2 2 2	Thm2,I
1 1 1 1 1 1 2 8 9	Thm2,IV	1 1 2 2 2 2 2 2 8	Thm2,II
1 1 1 1 1 1 4 4 4	Thm2,V	1 1 2 2 2 2 2 18 18	Thm2,III
1 1 1 1 1 1 5 5 9	Thm2,VI	1 1 2 2 2 2 4 16 18	Thm2,IV
1 1 1 1 1 1 9 9 9	Thm2,III	1 1 2 2 2 2 8 8 8	Thm2,V
1 1 1 1 1 1 9 9 18	Thm8	1 1 2 2 2 2 10 10 18	Thm2,VI
1 1 1 1 1 2 2 2 2	Thm9,16f	1 1 2 2 2 8 8 8 8	Thm2,VII
1 1 1 1 1 2 2 2 4	Thm9,16e	1 2 2 2 2 2 2 2 4	Thm9,16i
1 1 1 1 1 2 2 2 8	Thm9,16f	1 3 3 3 3 3 3 3 3	Thm10,VIII
1 1 1 1 1 2 2 8 9	Thm2,IV	2 2 2 2 2 2 2 2 8	Thm9,16i
1 1 1 1 1 2 3 3 3	Thm9,16g	2 2 2 2 2 2 2 4 4	Thm2,II
1 1 1 1 1 2 3 3 12	Thm9,16g	2 2 2 2 2 2 2 16 18	Thm2,IV
1 1 1 1 1 2 8 8 9	Thm2,IV	2 2 2 2 2 2 9 9 18	Thm2,III
1 1 1 1 1 2 8 9 9	Thm2,IV	2 2 2 2 2 4 4 8 8	Thm2,V
1 1 1 1 1 4 4 4 4	Thm2,V	2 2 2 2 2 4 8 8 18	Thm2,IV
1 1 1 1 1 5 5 5 9	Thm2,VI	2 2 2 2 2 4 9 9 16	Thm2,IV
1 1 1 1 1 5 5 9 9	Thm2,VI	2 2 2 2 2 5 5 10 18	Thm2,VI
1 1 1 1 2 2 2 2 2	Thm9,16h	2 2 2 2 2 9 9 10 10	Thm2,VI
1 1 1 1 2 2 2 2 4	Thm9,16f	2 2 2 2 4 4 8 8 8	Thm2,VII
1 1 1 1 2 2 2 2 8	Thm9,16g	3 3 3 3 3 3 3 3 3	Thm2,VIII
1 1 1 1 2 3 3 3 4	Thm9,16h	3 3 6 6 6 6 6 6 6	Thm2,VIII

Table 8 9-Variable designs in order 48

8−Variables								Ref	8−Variables								Ref
1	1	1	1	1	1	1	11	Thm9	1	1	1	2	2	2	8	10	Thm9
1	1	1	1	1	1	3	3	Thm9	1	1	1	2	2	3	3	4	Thm9
1	1	1	1	1	1	3	9	Thm9	1	1	1	2	2	3	3	8	Thm9
1	1	1	1	1	1	3	12	Thm9	1	1	1	2	2	3	3	9	Thm9
1	1	1	1	1	1	4	16	Thm9	1	1	1	2	2	3	6	8	Thm9
1	1	1	1	1	1	5	6	Thm9	1	1	1	2	2	3	8	15	Thm9
1	1	1	1	1	1	5	20	Thm9	1	1	1	2	2	4	4	4	Thm3
1	1	1	1	1	2	2	11	Thm9	1	1	1	2	2	4	4	8	Thm3
1	1	1	1	1	2	4	5	Thm9	1	1	1	2	2	4	8	8	Thm3
1	1	1	1	1	2	4	9	Thm9	1	1	1	2	2	4	8	9	Thm3
1	1	1	1	1	2	4	18	Thm9	1	1	1	2	2	4	9	16	Thm3
1	1	1	1	1	2	5	6	Thm9	1	1	1	2	2	5	5	9	Thm3
1	1	1	1	1	2	8	10	Thm9	1	1	1	2	2	5	5	10	Thm3
1	1	1	1	1	2	8	16	Thm3	1	1	1	2	2	5	9	10	Thm3
1	1	1	1	1	2	9	18	Thm3	1	1	1	2	2	9	10	10	Thm3
1	1	1	1	1	2	10	12	Thm9	1	1	1	2	3	3	3	8	Thm9
1	1	1	1	1	3	3	4	Thm9	1	1	1	2	3	3	3	9	Thm9
1	1	1	1	1	3	3	9	Thm9	1	1	1	2	3	3	3	10	Thm9
1	1	1	1	1	4	5	5	Thm9	1	1	1	2	3	3	4	9	Thm9
1	1	1	1	1	4	5	6	Thm9	1	1	1	2	3	3	9	12	Thm9
1	1	1	1	1	4	5	8	Thm9	1	1	1	2	4	4	4	8	Thm3
1	1	1	1	1	4	8	8	Thm3	1	1	1	2	4	4	8	8	Thm3
1	1	1	1	1	4	9	16	Thm3	1	1	1	2	4	8	8	8	Thm3
1	1	1	1	1	5	6	8	Thm9	1	1	1	3	3	3	3	9	Thm9
1	1	1	1	1	9	10	10	Thm3	1	1	2	2	2	2	2	9	Thm9
1	1	1	1	2	2	2	11	Thm9	1	1	2	2	2	2	3	8	Thm9
1	1	1	1	2	2	3	3	Thm9	1	1	2	2	2	2	3	9	Thm9
1	1	1	1	2	2	3	9	Thm9	1	1	2	2	2	2	3	12	Thm9
1	1	1	1	2	2	3	12	Thm9	1	1	2	2	2	2	4	5	Thm9
1	1	1	1	2	2	4	18	Thm9	1	1	2	2	2	2	5	6	Thm9
1	1	1	1	2	2	5	8	Thm9	1	1	2	2	2	2	5	9	Thm9
1	1	1	1	2	2	8	8	Thm9	1	1	2	2	2	2	8	9	Thm3
1	1	1	1	2	2	10	12	Thm9	1	1	2	2	2	2	8	10	Thm2
1	1	1	1	2	3	3	8	Thm9	1	1	2	2	2	2	9	9	Thm3
1	1	1	1	2	3	3	9	Thm9	1	1	2	2	2	2	9	16	Thm3
1	1	1	1	2	3	6	9	Thm9	1	1	2	2	2	2	9	18	Thm3
1	1	1	1	2	3	9	15	Thm9	1	1	2	2	2	2	10	12	Thm9
1	1	1	1	2	4	4	8	Thm9	1	1	2	2	2	3	3	8	Thm9
1	1	1	1	2	4	4	16	Thm9	1	1	2	2	2	4	4	4	Thm3
1	1	1	1	2	5	9	10	Thm3	1	1	2	2	2	4	4	8	Thm3
1	1	1	1	3	3	3	9	Thm9	1	1	2	2	2	4	4	16	Thm2
1	1	1	1	3	3	5	9	Thm9	1	1	2	2	2	4	8	8	Thm3
1	1	1	1	4	4	4	16	Thm9	1	1	2	2	2	4	8	9	Thm3
1	1	1	2	2	2	2	11	Thm9	1	1	2	2	2	4	9	16	Thm3
1	1	1	2	2	2	3	9	Thm9	1	1	2	2	2	5	5	9	Thm3
1	1	1	2	2	2	4	5	Thm9	1	1	2	2	2	5	5	10	Thm3
1	1	1	2	2	2	4	8	Thm9	1	1	2	2	2	5	9	10	Thm3
1	1	1	2	2	2	5	6	Thm9	1	1	2	2	2	9	10	10	Thm3

Table 9 8–Variable designs in order 48

8−Variables	Ref
1 1 2 2 3 3 3 8	Thm9
1 1 2 2 3 3 3 9	Thm9
1 1 2 2 3 3 4 8	Thm9
1 1 2 2 3 3 8 12	Thm9
1 1 2 2 4 4 4 4	Thm3
1 1 2 2 4 4 8 8	Thm3
1 1 2 3 3 3 4 5	Thm9
1 1 2 3 3 3 5 6	Thm9
1 1 2 3 3 3 5 8	Thm9
1 1 2 3 3 3 5 9	Thm9
1 1 2 3 3 3 9 10	Thm9
1 2 2 2 2 2 2 11	Thm9
1 2 2 2 2 2 4 5	Thm9
1 2 2 2 2 2 5 6	Thm9
1 2 2 2 2 2 5 8	Thm9
1 2 2 2 2 2 8 9	Thm3
1 2 2 2 2 2 8 10	Thm9
1 2 2 2 2 2 8 16	Thm3
1 2 2 2 2 2 9 9	Thm3
1 2 2 2 2 2 9 16	Thm3
1 2 2 2 2 2 9 18	Thm3
1 2 2 2 2 3 3 4	Thm9
1 2 2 2 2 3 3 9	Thm9
1 2 2 2 2 4 4 4	Thm3
1 2 2 2 2 4 4 8	Thm3
1 2 2 2 2 4 8 8	Thm3
1 2 2 2 2 4 8 9	Thm3
1 2 2 2 2 4 9 16	Thm3
1 2 2 2 2 5 5 9	Thm3
1 2 2 2 2 5 5 10	Thm3
1 2 2 2 2 5 9 10	Thm3
1 2 2 2 2 9 10 10	Thm3

8−Variables	Ref
1 2 2 2 3 3 3 8	Thm9
1 2 2 2 4 4 4 4	Thm3
1 2 2 2 4 4 4 8	Thm3
1 2 2 2 4 4 8 8	Thm3
1 2 2 2 4 8 8 8	Thm3
1 2 2 3 3 3 5 8	Thm9
1 4 4 4 4 4 4 4	Thm12
2 2 2 2 2 2 2 10	Thm9
2 2 2 2 2 2 3 8	Thm9
2 2 2 2 2 2 3 9	Thm9
2 2 2 2 2 2 4 8	Thm9
2 2 2 2 2 2 4 16	Thm2
2 2 2 2 2 2 5 8	Thm9
2 2 2 2 2 2 8 9	Thm3
2 2 2 2 2 2 8 10	Thm9
2 2 2 2 2 2 9 16	Thm3
2 2 2 2 2 4 4 10	Thm2
2 2 2 2 2 4 8 9	Thm3
2 2 2 2 2 5 5 9	Thm3
2 2 2 2 2 5 9 10	Thm3
2 2 2 2 2 5 9 10	Thm3
2 2 2 2 3 3 16 18	Thm2
2 2 2 2 4 4 4 4	Thm3
2 2 2 2 4 4 4 8	Thm3
2 2 2 2 4 4 4 8	Thm3
2 2 2 2 6 6 10 18	Thm2
3 3 3 3 3 3 6 6	Thm3
3 3 3 3 3 6 6 6	Thm3
3 3 3 3 6 6 6 6	Thm3
3 3 3 6 6 6 6 6	Thm3
5 5 5 5 5 5 5 5	Thm12

Table 9(cont) 8–Variable designs in order 48

7−Variables	Ref
1 1 1 1 2 6 18	Thm9
1 1 1 1 4 6 18	Thm9
1 1 1 1 6 6 24	Thm9
1 1 1 2 4 10 10	Thm9
1 1 1 3 4 4 16	Thm9
1 1 1 4 4 6 6	Thm9
1 1 2 2 5 5 8	Thm9
1 1 2 3 3 3 3	Thm9
1 2 2 2 3 6 8	Thm9
1 2 2 3 3 3 8	Thm9

7−Variables	Ref
1 2 2 3 3 4 8	Thm9
1 2 2 3 3 5 8	Thm9
1 2 3 3 3 3 4	Thm9
2 2 2 2 2 6 8	Thm9
2 2 2 2 3 5 8	Thm9
2 2 2 3 3 3 8	Thm9
2 2 2 3 3 3 9	Thm9
2 2 2 3 3 4 8	Thm9
2 2 2 4 8 10 10	Thm9
2 2 3 3 3 3 8	Thm9

Table 10 7-Variable designs in order 48

6-Variables	Ref	6-Variables	Ref
1 1 1 2 4 10	Thm3,XI	1 2 2 2 4 10	Thm2,XI
1 1 1 3 4 9	Thm3,XII	1 2 2 2 8 10	Thm2,XI
1 1 1 3 4 18	Thm2,XII	1 2 2 2 8 18	Thm3,IV
1 1 1 4 4 20	Thm9	1 2 2 2 8 20	Thm2,XI
1 1 1 4 5 20	Thm9	1 2 2 2 9 16	Thm3,IV
1 1 1 4 6 6	Thm3,XIII	1 2 2 2 9 18	Thm2,III
1 1 1 4 6 9	Thm2,XII	1 2 2 3 4 9	Thm2,XII
1 1 1 4 6 18	Thm2,XII	1 2 2 3 4 18	Thm2,XII
1 1 1 4 8 20	Thm2,XI	1 2 2 3 8 9	Thm2,XII
1 1 1 6 8 10	Thm2,XII	1 2 2 3 8 18	Thm2,XII
1 1 1 6 8 20	Thm2,XII	1 2 2 4 4 10	Thm2,XI
1 1 1 11 11 11	Thm7	1 2 2 4 4 20	Thm2,XI
1 1 2 2 8 20	Thm2,XI	1 2 2 4 6 9	Thm2,XII
1 1 2 2 8 25	Thm9	1 2 2 4 6 18	Thm2,XIV
1 1 2 2 9 18	Thm2,III	1 2 2 4 8 10	Thm2,XI
1 1 2 3 3 6	Thm7	1 2 2 4 8 18	Thm3,IV
1 1 2 3 4 9	Thm2,XII	1 2 2 4 8 20	Thm2,XI
1 1 2 3 4 18	Thm2,XII	1 2 2 4 9 9	Thm2,III
1 1 2 3 8 18	Thm2,XII	1 2 2 4 9 16	Thm3,IV
1 1 2 4 4 10	Thm2,XI	1 2 2 4 9 18	Thm2,III
1 1 2 4 4 20	Thm2,XI	1 2 2 5 8 15	Thm9
1 1 2 4 6 9	Thm2,XII	1 2 2 6 8 9	Thm2,XIV
1 1 2 4 6 18	Thm2,XII	1 2 2 6 8 10	Thm2,XIII
1 1 2 4 8 10	Thm2,XI	1 2 2 6 8 18	Thm2,XIV
1 1 2 4 8 20	Thm2,XI	1 2 2 6 8 20	Thm2,XIII
1 1 2 4 9 9	Thm2,III	1 2 2 8 10 15	Thm9
1 1 2 4 9 18	Thm2,III	1 2 2 8 12 15	Thm9
1 1 2 6 6 12	Thm7	1 2 3 4 4 9	Thm2,XIV
1 1 2 6 8 9	Thm2,XIV	1 2 3 4 4 18	Thm2,XIV
1 1 2 6 8 10	Thm2,XII	1 2 3 4 8 9	Thm2,XIV
1 1 2 6 8 18	Thm2,XIV	1 2 3 4 8 18	Thm2,XIV
1 1 2 6 8 20	Thm2,XII	1 2 4 4 6 9	Thm2,XIV
1 1 2 9 9 18	Thm7	1 2 4 4 6 18	Thm2,XIV
1 1 2 11 11 22	Thm7	1 2 4 4 8 9	Thm3,IV
1 1 3 4 4 9	Thm2,XIV	1 2 4 4 8 18	Thm3,IV
1 1 3 4 4 18	Thm2,XIV	1 2 4 4 9 9	Thm2,III
1 1 3 4 8 9	Thm2,XIV	1 2 4 4 9 16	Thm3,IV
1 1 3 4 8 18	Thm2,XIV	1 2 4 4 9 18	Thm2,III
1 1 3 4 9 20	Thm9	1 2 4 4 18 18	Thm2,III
1 1 3 5 9 15	Thm9	1 2 4 6 8 9	Thm2,XIV
1 1 3 5 9 24	Thm9	1 2 4 6 8 18	Thm2,XIV
1 1 3 6 6 9	Thm9	1 3 3 5 9 12	Thm9
1 1 3 9 12 15	Thm9	1 4 4 4 8 9	Thm3,IV
1 1 4 4 6 9	Thm2,XIV	1 4 4 4 8 18	Thm3,IV
1 1 4 4 6 18	Thm2,XIV	1 4 4 4 9 16	Thm3,IV
1 1 4 4 9 9	Thm2,III	2 2 2 2 8 9	Thm3,IV
1 1 4 4 9 18	Thm2,III	2 2 2 2 9 16	Thm3,IV
1 1 4 6 8 9	Thm2,XIV	2 2 2 3 4 9	Thm2,XIII
1 1 4 6 8 18	Thm2,XIV	2 2 2 3 4 18	Thm2,XIII
1 1 5 6 6 24	Thm9	2 2 2 3 8 9	Thm2,XIII

Table 11 6-Variable designs in order 48

6-Variables	Ref	6-Variables	Ref
2 2 2 3 8 18	Thm2,XIII	2 2 4 6 6 12	Thm7
2 2 2 4 6 9	Thm2,XIII	2 2 4 6 8 9	Thm2,XIV
2 2 2 4 6 18	Thm2,XIII	2 2 4 6 8 18	Thm2,XIV
2 2 2 4 8 9	Thm3,IV	2 2 4 7 7 14	Thm7
2 2 2 4 8 18	Thm3,IV	2 2 5 8 12 15	Thm9
2 2 2 4 9 9	Thm2,III	2 3 5 5 6 9	Thm9
2 2 2 4 9 16	Thm3,IV	2 3 5 5 9 12	Thm9
2 2 2 4 9 18	Thm2,III	2 4 4 4 8 9	Thm3,IV
2 2 2 5 5 5	Thm7	2 4 4 4 8 18	Thm3,IV
2 2 2 6 6 6	Thm7	2 4 4 4 9 16	Thm3,IV
2 2 2 6 8 9	Thm2,XIV	3 3 3 4 4 4	Thm7
2 2 2 6 8 10	Thm2,XIII	3 3 3 5 5 9	Thm9
2 2 2 6 8 18	Thm2,XIV	3 3 3 7 7 7	Thm7
2 2 2 6 8 20	Thm2,XIII	3 3 3 8 8 8	Thm7
2 2 2 7 7 7	Thm7	3 3 4 4 6 8	Thm7
2 2 3 3 4 6	Thm7	3 3 6 7 7 14	Thm7
2 2 3 4 4 9	Thm2,XIV	3 3 6 8 8 16	Thm7
2 2 3 4 4 18	Thm2,XIV	3 3 6 9 9 18	Thm7
2 2 3 4 8 9	Thm2,XIV	4 4 4 5 5 5	Thm7
2 2 3 4 8 18	Thm2,XIV	4 4 4 6 6 6	Thm7
2 2 3 5 8 15	Thm9	4 4 5 5 8 10	Thm7
2 2 4 4 6 9	Thm2,XIV	4 4 6 6 8 12	Thm7
2 2 4 4 6 18	Thm2,XIV	4 4 8 8 8 16	Thm7
2 2 4 4 8 9	Thm3,IV	5 5 5 6 6 6	Thm7
2 2 4 4 9 9	Thm2,III	5 5 5 7 7 7	Thm7
2 2 4 4 9 16	Thm3,IV	5 5 6 6 10 12	Thm7
2 2 4 4 9 18	Thm2,III	5 5 7 7 10 14	Thm7
2 2 4 5 5 10	Thm7		

Table 11 (cont) 6-Variable designs in order 48

5−Variables	Ref	5−Variables	Ref
1 3 3 3 3	Thm7	2 10 10 10 10	Thm7
1 5 5 5 5	Thm7	3 3 3 3 4	Thm7
1 6 6 6 6	Thm7	3 3 3 3 7	Thm7
1 9 9 9 9	Thm7	3 3 3 3 8	Thm7
1 10 10 10 10	Thm7	3 7 7 7 7	Thm7
1 11 11 11 11	Thm7	3 8 8 8 8	Thm7
2 2 2 2 7	Thm7	3 9 9 9 9	Thm7
2 3 3 3 3	Thm7	4 6 6 6 6	Thm7
2 5 5 5 5	Thm7	5 5 5 5 6	Thm7
2 6 6 6 6	Thm7	5 5 5 5 7	Thm7
2 7 7 7 7	Thm7	5 6 6 6 6	Thm7
2 9 9 9 9	Thm7	5 7 7 7 7	Thm7

Table 12 5-Variable designs in order 48