New large sets of t-designs with prescribed automorphism groups

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Abstract

In this paper, we investigate the existence of large sets of 3-designs of prime sizes with prescribed automorphism groups $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$ for $q < 60$. We also construct some new interesting large sets by the use of the computer program DISCRETA. The results obtained through these direct methods along with known recursive constructions are combined to prove more extensive theorems on the existence of large sets.

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1 Introduction

Let t, k, v and λ be integers such that $0 < t \leq k \leq v$ and $\lambda > 0$. Let X be a v-set and $P_k(X)$ denote the set of all k-subsets of X. A *simple* t-(v, k, λ) *design* is a pair $(X, \mathcal{B})$ in which $\mathcal{B}$ is a subset of $P_k(X)$ such that every t-subset of X appears in exactly λ elements of $\mathcal{B}$.

The central question concerning t-designs is the existence problem. Two different approaches have been developed to tackle this problem. These are direct and recursive constructions. The direct constructions by the use of prescribed automorphism groups have already yielded several infinite series of designs. The objective of recursive constructions is to find in a more comprehensive way lots of infinite series of designs. Combining these two approaches provides a powerful tool to obtain existence results.

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The recursive constructions rely on the notion of *large sets of t-designs*. Let $N > 1$. A *large set* of t-(v,k,λ) designs of size N, denoted by $\mathrm{LS}[N](t,k,v)$, is a set of N disjoint t-(v,k,λ) designs $(X,\mathcal{B}_i)$ such that $\{\mathcal{B}_i|\ 1 \le i \le N\}$ is a partition of $P_k(X)$. Note that we have $N = \binom{v-t}{k-t}/\lambda$. A set of well known necessary conditions for the existence of an $\mathrm{LS}[N](t,k,v)$ is

$$N \Big| \binom{v-i}{k-i}, \qquad 0 \le i \le t. \tag{1.1}$$

There have been several papers on large sets. We refer the reader to [7, 10, 15] and the references therein.

The direct methods prescribe a group of automorphisms of the desired design that need not be its full automorphism group. There may be several designs admitting the same prescribed group of automorphisms. Some of them may even form a large set of t-designs. Then the large set is uniform with respect to this group.

In this paper, we directly construct uniform large sets of 3-designs, mainly using as prescribed automorphism groups $\mathrm{PSL}(2,q)$ and $\mathrm{PGL}(2,q)$ for odd prime powers $q < 60$. The method is to first determine the number of orbits of different sizes of such groups and then to combine these orbits to 3-designs that all have the same parameters. This combination is done by solving systems of Diophantine equations. We then base the recursive constructions on these designs to obtain some extensive results on the existence of large sets. In some cases we have to add some sporadic results for other types of groups and for higher t obtained by DISCRETA, a program developed at the University of Bayreuth. In particular, we find large sets for $t = 6$ with automorphism group $\mathrm{P\Gamma L}(2,32)$ using DISCRETA. Thus, we also consider characteristic 2 in this case. We summarize the main results of this paper in the following theorems.

Theorem 1.1 *Let $k < 81$ or $2 \cdot 3^{n-1} + 3^{n-3} \le k < 3^n$ for some integer $n > 4$. Then there exists an LS$[3](3,k,v)$ if and only if the necessary conditions* (1.1) *hold.*

Theorem 1.2 *Let $k < 29$. Then there exists an LS$[29](2,k,v)$ if and only if the necessary conditions* (1.1) *hold.*

2 Orbits from $\mathrm{PSL}(2,q)$ and $\mathrm{PGL}(2,q)$

Let $q = p^n$ be an odd prime power and let $G = \mathrm{PSL}(2,q)$ if $q \equiv 3 \pmod 4$ and $G = \mathrm{PGL}(2,q)$, otherwise. The group G is 3-homogeneous on the projective line and therefore it may be used to construct 3-designs and large sets of 3-designs. It is easy to see that a set of k-subsets of the projective line is the block set of a 3-$(q+1,k,\lambda)$ design with automorphism group G for some λ if and only if it is a union of orbits of G.

Table 1: The orbit numbers from PSL(2, 83)

Stabilizer orders	$k = 27$	$k = 28$	$k = 29$	$k = 30$
1	26273404505520999	53485144257137502	103281659090991984	189349707158658344
2		1258518732		2349341465
3	246675			467610
4		38757		
6				2145
7		34		
14		13		
28		3		

In [5, 6], several formulae for the numbers of orbits of different sizes from the action of G on k-subsets of the projective line when $k \not\equiv 0, 1 \pmod p$ are given. We used a computer program to evaluate those formulae and compute orbit numbers for $q < 60$ and $k \not\equiv 0, 1 \pmod p$. For $q = 25, 27$, $k \equiv 0, 1 \pmod p$ and $q = 49, k = 7, 8$, we determined the orbit numbers by the use of DISCRETA. Hence, the orbit numbers for all $q < 60$ and k except when $q = 49, k = 14, 15, 21, 22$ are determined. We use these results to obtain large sets of 3-designs of prime sizes. In the next section we describe an algorithm for combining orbits to large sets. The results from an implementation of this algorithm are listed in Table 2. In this table, for any $q < 60$ and $k < q/2$, we give many prime numbers p for which LS[p]$(3, k, q + 1)$ are constructible from combining the orbits of G on k-subsets of the projective line. Note that only those values of k for which large sets were found appear in the table.

The large sets LS[3]$(3, k, 84)$ for $k = 27, 28, 29, 30$ are of special interest since they can be used in the recursive constructions to obtain strong existence results. This is done in Section 5. Here, we show that these large sets exist. In Table 1, the numbers of orbits for all possible stabilizer sizes from the action of PSL(2, 83) on k-subsets of the projective line for $k = 27, 28, 29, 30$ are given. It is easily seen that the numbers are divisible by 3 or can be made so by a small tradeoff between orbits of different sizes and therefore LS[3]$(3, k, 84)$ are constructed by a suitable partitioning of orbits.

3 Combining orbits to large sets

Let G be a t-homogeneous permutation group on a finite set X and let G act on k-subsets of X. Suppose that there are $s_1, s_2, \ldots, s_n$ orbits of size $|G|/q_1, |G|/q_2, \ldots, |G|/q_n$, respectively,

Table 2: Large sets from PSL$(2,q)$ and PGL$(2,q)$ for $q<60$

q	$(k;\{N_1,N_2,...,N_r\})$
11	$(6;\{2\})$
19	$(6;\{2\})$ $(7;\{2\})$ $(8;\{2\})$
23	$(5;\{7\})$ $(7;\{3,19\})$ $(8;\{3\})$ $(9;\{2\})$ $(10;\{2,17,19\})$ $(12;\{2\})$
25	$(12;\{2\})$
27	$(5;\{2,5\})$ $(6;\{2\})$ $(8;\{5\})$ $(11;\{5,19,23\})$ $(12;\{5\})$
29	$(10;\{3\})$ $(11;\{3,5,13,23\})$ $(12;\{3,5\})$ $(13;\{3,5\})$
31	$(8;\{3\})$ $(9;\{3,5,13,29\})$ $(10;\{3,13,23,29\})$ $(11;\{3,5,13,23,29\})$ $(12;\{3,7,13,23,29\})$ $(13;\{3,5,7,23,29\})$ $(14;\{3,5,19,23,29\})$ $(15;\{3,19,23,29\})$ $(16;\{3\})$
37	$(7;\{2\})$ $(8;\{2\})$ $(12;\{2,3,7,17,29,31\})$ $(13;\{2,3,7,17,29,31\})$ $(14;\{2,3,5,17\})$ $(15;\{2,3,5,17,29,31\})$ $(16;\{2,3,5,17,23,29,31\})$ $(17;\{2,3,5,11,23,29,31\})$ $(18;\{2,3,5,7,11,23,29,31\})$
41	$(13;\{2,17,19,31\})$ $(15;\{2\})$ $(16;\{2\})$ $(17;\{2\})$ $(19;\{2,3,5,13,29,31\})$ $(20;\{2\})$
43	$(6;\{2\})$ $(10;\{13,19\})$ $(12;\{13,17,19\})$ $(13;\{2,17,19,41\})$ $(16;\{2\})$ $(17;\{2,19,29,31,41\})$ $(18;\{2,3,19,29,31,41\})$ $(19;\{2,3,13,29,31,41\})$ $(20;\{2,3,5,13,29,31,41\})$
47	$(4;\{3,5\})$ $(5;\{3,11\})$ $(6;\{3,11,43\})$ $(7;\{3,11,43\})$ $(8;\{3\})$ $(9;\{5,11\})$ $(10;\{11,13,41,43\})$ $(11;\{13,19,41,43\})$ $(12;\{13,19\})$ $(13;\{3,19,37,41,43\})$ $(14;\{3,5,19,41,43\})$ $(15;\{3,17,19,41,43\})$ $(16;\{3\})$ $(17;\{2,3,11,19,37,41,43\})$ $(18;\{2,11,19,31,41,43\})$ $(19;\{2,5,11,31,37,41,43\})$ $(20;\{2,11,29,31,41,43\})$ $(21;\{2,11,29,31,41,43\})$ $(22;\{2,3,29,31,41,43\})$ $(24;\{2\})$
49	$(13;\{19\})$ $(14;\{?\})$ $(15;\{?\})$ $(19;\{2,11,23,41,43,47\})$ $(21;\{?\})$ $(22;\{?\})$ $(23;\{2,29,31,41,43,47\})$ $(24;\{2,3\})$
53	$(7;\{2,5,7,17\})$ $(8;\{2,5,7,17,47\})$ $(10;\{2,5,7,17,23,47\})$ $(11;\{2,5,7,17,23,47\})$ $(12;\{2\})$ $(16;\{5,7\})$ $(17;\{5,7,19,23,41,43,47\})$ $(19;\{5,7,23,41,43,47\})$ $(20;\{5,7\})$ $(21;\{5,7,17,23,41,43,47\})$ $(22;\{5,7\})$ $(23;\{2,5,7,17,41,43,47\})$ $(24;\{2,5,7,17,31,41,47\})$ $(25;\{2,7,17,31,41,43,47\})$
59	$(6;\{2,7\})$ $(7;\{2,3,11,19\})$ $(8;\{3,11,19\})$ $(9;\{3,11,13,19,53\})$ $(10;\{3\})$ $(11;\{3,5,13,17,19,53\})$ $(12;\{3,5,7,13,17,19,53\})$ $(13;\{2,3,5,7,17,19,53\})$ $(14;\{2,3,5,7,17,19,47,53\})$ $(15;\{2,3,5,7,17,19,23,47,53\})$ $(16;\{3,5,7,17,19,23,47,53\})$ $(17;\{3,5,7,11,19,23,47,53\})$ $(18;\{3,5\})$ $(19;\{3,5,7,11,23,43,47,53\})$ $(20;\{3,5,7,11,23,41,43,47,53\})$ $(21;\{2,3,5,7,11,23,41,43,47,53\})$ $(22;\{2,3,5,7,13,23,41,43,47,53\})$ $(23;\{2,3,5,7,13,19,41,43,47,53\})$ $(25;\{3,7,13,41\})$ $(26;\{3,7,29,41,43,47,53\})$ $(27;\{7,17,19,41,43,47,53\})$ $(28;\{7,41\})$

where $0 < q_1 < q_2 < \cdots < q_n$. We combine the orbits to large sets of t-designs. We do this by combining the orbits to sets of sizes $|G|/q_1$, $|G|/q_2, \ldots, |G|/q_n$ such that for $1 \leq i \leq n$ the sets of size $|G|/q_i$ can be partitioned into N (possibly combined) orbits of size $|G|/q_i$.

Example Let $G = \mathrm{PSL}(2, 27)$, $t = 3$ and $k = 8$. Then there are 284 orbits of size $|G|$, 61 orbits of size $|G|/2$ and 7 orbits of size $|G|/4$. We have the following solution for $N = 5$:

	$\lvert G\rvert/1$	$\lvert G\rvert/2$	$\lvert G\rvert/4$	$\cup$
$\lvert G\rvert/1$	284	22	0	295
$\lvert G\rvert/2$		39	2	40
$\lvert G\rvert/4$			5	5
	284	61	7	

It means that we combine 284 orbits of size $|G|$ and 22 orbits of size $|G|/2$ to 295 sets of size $|G|$. These 295 sets of size $|G|$ can be partitioned into 5 sets, each consisting of 59 unions of orbits of size $|G|$. We take 39 orbits of size $|G|/2$ and 2 orbits of size $|G|/4$ to get 40 sets of size $|G|/2$. These 40 orbits can be partitioned into 5 sets, each consisting of 8 unions of orbits of size $|G|/2$. And finally we have 5 orbits of size $|G|/4$. Therefore, we have an $\mathrm{LS}[5](3, 8, 28)$.

We use arithmetic rules to combine several shorter orbits. For example, 2 orbits of size $|G|/2$ give one set of size $|G|$. In the first step of our method we compute all possible rules to combine orbits, i.e. for $1 \leq i \leq n$ we compute all solutions of

$$|G|/q_i = \sum_{j=1}^{n} z_{i,j}|G|/q_j \quad \text{and } \mathrm{z}_{i,j} \in \mathbb{Z},\ z_{ij} \geq 0. \tag{3.1}$$

Each solution $(z_{i,1}^{(k)}, z_{i,2}^{(k)}, \ldots, z_{i,n}^{(k)})$, $1 \leq i \leq n$, $1 \leq k \leq m_i$ of the above equations is called rule. Note that $z_{i,j}^{(k)} = 0$ for $j < i$ and $m_n = 1$. Then in the second step of our method we use the rules to combine orbits.

Theorem 3.1 *The solutions of the system of linear equations*

$$\begin{pmatrix}
1 & \ldots & 1 & & & & & & -N & & \\
 & & & 1 & \ldots & 1 & & & & -N & \\
 & & & & & & \ddots & & & & \ddots \\
 & & & & & & & 1 & & & & -N \\
z_{1,1}^{(1)} & \ldots & z_{1,1}^{(m_1)} & z_{2,1}^{(1)} & \ldots & z_{2,1}^{(m_2)} & \ldots & z_{n,1}^{(1)} & & & \\
z_{1,2}^{(1)} & \ldots & z_{1,2}^{(m_1)} & z_{2,2}^{(1)} & \ldots & z_{2,2}^{(m_2)} & \ldots & z_{n,2}^{(1)} & & & \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & & & \\
z_{1,n}^{(1)} & \ldots & z_{1,n}^{(m_1)} & z_{2,n}^{(1)} & \ldots & z_{2,n}^{(m_2)} & \ldots & z_{n,n}^{(1)} & & &
\end{pmatrix} \cdot \begin{pmatrix} x_1^{(1)} \\ x_1^{(2)} \\ \vdots \\ x_1^{(m_1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \\ y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix},$$

where for $1 \le i \le n$, $1 \le j \le n$, $1 \le k \le m_i$ *the variables* $x_{i,j}^{(k)}, y_i \in \mathbb{Z}$ *and*

$$0 \le x_{i,j}^{(k)} \le \lfloor s_i / \max\{z_{i,j}^{(k)} \mid 1 \le j \le n\} \rfloor,$$

correspond to large sets of size N *with automorphism group* G.

Proof The first n equations ensure that the number of sets of size $|G|/q_i$ for $1 \le i \le n$ can be partitioned into N sets. The second n equations ensure that all orbits are used. □

The solutions of 3.1 and the system of Diophantine linear equations in Theorem 3.1 can be solved for example with the methods described in [21].

The existence of a solution in Theorem 3.1 is not necessary for the existence of large sets of size N with automorphism group G. But the restriction of the search to large sets of this form in many cases reduces the sizes of the resulting systems of Diophantine linear equations. The size of the system then only depends on the number of rules and not on the size of N. This enables the construction of large sets in cases where the general approach – partitioning the set of orbits into N designs – seems to fail. This is a bin-packing problem which is NP-hard. So, appropriate restrictions on the space of solutions will be essential for reaching larger cases. Nevertheless, we give the system of Diophantine linear equations which we used to construct large sets in this case:

$$\begin{pmatrix}
\frac{|G|}{q_1} & \frac{|G|}{q_2} & \cdots & \frac{|G|}{q_n} & & & & & & & & & & \\
& & & & \frac{|G|}{q_1} & \frac{|G|}{q_2} & \cdots & \frac{|G|}{q_n} & & & & & & \\
& & & & & & & & \ddots & & & & & \\
& & & & & & & & & \frac{|G|}{q_1} & \frac{|G|}{q_2} & \cdots & \frac{|G|}{q_n} \\
1 & & & & 1 & & & & & 1 & & & \\
& 1 & & & & 1 & & & & & 1 & & \\
& & \ddots & & & & \ddots & & & & & \ddots & \\
& & & 1 & & & & 1 & & & & & 1
\end{pmatrix} \cdot \begin{pmatrix} x_{1,1} \\ \vdots \\ x_{1,n} \\ x_{2,1} \\ \vdots \\ x_{2,n} \\ \vdots \\ x_{N,1} \\ \vdots \\ x_{N,n} \end{pmatrix} = \begin{pmatrix} \binom{n}{k}/N \\ \binom{n}{k}/N \\ \vdots \\ \binom{n}{k}/N \\ s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix},$$

where $x_{i,j} \in \mathbb{Z}$, $x_{i,j} > 0$, $1 \le i, j \le n$. The matrix has $n \cdot N$ columns and $N + n$ rows. We only tried to solve this system of equations if the system from Theorem 3.1 had no solutions.

4 Sporadic large sets

The computer system DISCRETA is used to prescribe any group of automorphisms to construct t-designs. It employs the method of Kramer and Mesner [13] that turns the construction problem

Table 3: Large sets of size 3 found using DISCRETA

Group	Large set
$\mathrm{Hol}(\mathrm{C}_{23})$	$\mathrm{LS}[3](5,6,23)$
$\mathrm{PSL}(2,23)$	$\mathrm{LS}[3](5,8,24)$
$\mathrm{P\Gamma L}(2,32)$	$\mathrm{LS}[3](5,7,33)$
$\mathrm{P\Gamma L}(2,32)$	$\mathrm{LS}[3](6,10,33)$

into the problem of solving a Diophantine system of linear equations by a vector with entries from $\{0,1\}$ only. Large sets may be obtained by stepwise deletion of columns that belong to orbits which already have been used for designs in building up the large set. This has been described in more detail in [15].

Prescribing $\mathrm{P\Gamma L}(2,33)$ we this way could construct an $\mathrm{LS}[3](6,10,33)$. The size of the Kramer-Mesner matrix is 53×2964. We do not include here the matrix and the lists of about 1000 columns that yield the individual designs of this large set. The details may be obtained from the authors. This large set yields further large sets $\mathrm{LS}[3](5,9,32)$, $\mathrm{LS}[3](5,10,32)$, $\mathrm{LS}[3](4,8,31)$, $\mathrm{LS}[3](4,9,31)$, $\mathrm{LS}[3](4,10,31)$ and $\mathrm{LS}[3](3,k,30)$ for $7 \leq k \leq 10$ by iteratively taking derived and residual designs of the designs in the large set (see Theorem 5.1). There are further large sets $\mathrm{LS}[3](5,k,v)$ that we could construct using DISCRETA. We list the results in Table 3.

New sporadic $\mathrm{LS}[N](t,k,v)$ for primes $N > 3$ were constructed for $t \leq 3$ and notably also an $\mathrm{LS}[13](6,10,33)$ was found. The results are shown in Table 4. Here is an explanation of the notation in the table. The group $G-$ denotes the stabilizer of a point in G. Adding a fixed point to a group G results in a group denoted as $G+$. The holomorph of a group is the semidirect product of that group with its automorphism group. In case of a prime p the automorphism group of C_p is cyclic. There is exactly one subgroup of an order d in a cyclic group where d divides the group order. $\mathrm{Hol}(\mathrm{C}_p)^i$ denotes the semidirect product of C_p with the unique automorphism group of index i in the full automorphism group.

5 Recursive constructions

In this section we use the results of the previous sections along with the known recursive constructions to find several new results on the existence of large sets. We first introduce some notation. Let m and n be positive integers. We denote the quotient and the remainder of division m by n by $[m/n]$ and (m/n), respectively. Let N, t and k be natural numbers. The set

Table 4: Large sets of prime sizes found using DISCRETA

Group	Large set
$\mathrm{Hol}(C_{17})^4$	$LS[5](2,8,17)$
$PSL(2,23)-$	$LS[5](3,9,23)$
$D_7 \times D_4$	$LS[5](3,6,28)$
$PSL(2,32)$	$LS[5](3,9,33)$
D_{17}	$LS[7](3,5,17)$
D_{17}	$LS[7](3,6,17)$
$\mathrm{Hol}(C_{23})^2+$	$LS[7](3,4,24)$
$\mathrm{Hol}(C_{23})^2+$	$LS[7](3,6,24)$
$PSL(2,32)$	$LS[7](3,13,33)$
C_{15}	$LS[13](2,4,15)$
C_{15}	$LS[13](2,7,15)$
$PSL(2,32)$	$LS[13](3,12,33)$
$P\Gamma L(2,32)$	$LS[13](6,10,33)$
C_{19}	$LS[17](2,4,19)$
$\mathrm{Hol}(C_{19})^3$	$LS[17](2,6,19)$
$\mathrm{Hol}(C_{19})^3$	$LS[17](2,7,19)$
C_{21}	$LS[19](2,4,21)$
C_{25}	$LS[23](2,4,25)$
$AGL(1,31)$	$LS[29](2,7,31)$
$PGL(2,32)-$	$LS[29](3,6,32)$
$\mathrm{Hol}(C_{19})$	$LS[13 \cdot 17](2,8,19)$
$PSL(2,31)$	$LS[3 \cdot 7](3,12,32)$

of all v for which an LS$[N](t,k,v)$ exists is denoted by A$[N](t,k)$. The set of all v which satisfy the necessary conditions (1.1) is denoted by B$[N](t,k)$. In the following we review a few useful theorems and recursive constructions which are utilized to find some new existence results.

Theorem 5.1 **[2, 11]** *If there exists an LS$[N](t,k,v)$, then there exist LS$[N](t-i,k-j,v-l)$ for all $0 \le j \le l \le i \le t$.*

Theorem 5.2 **[11]** *Let p^α be a prime power. $v \in B[p^\alpha](t,k)$ if and only if there exist distinct positive integers ℓ_i $(1 \le i \le \alpha)$ such that $t \le (v/p^{\ell_i}) < (k/p^{\ell_i})$.*

Theorem 5.3 **[3]** *If LS$[N](t,i,v)$ exist for all $t+1 \le i \le k$ and an LS$[N](t,k,u)$ also exists, then LS$[N](t,k,u+l(v-t))$ exist for all $l \ge 1$.*

Theorem 5.4 *If there exist LS$[N](t,k,v)$ and LS$[N](t,k+1,v)$, then there exists an LS$[N](t,k+1,v+1)$.*

Corollary 5.1 *If LS$[N](t,i,v)$ exist for all $t+1 \le i \le k$, then LS$[N](t,i,l(v-t)+j)$ exist for all $l \ge 1$, $t+1 \le i \le k$ and $t \le j < i$.*

Theorem 5.5 **[11]** *Let p be an odd prime and let t,k and s be nonnegative integers such that $p^s - 1 \le t < p^{s+1} - 1$ and $t < k$. Suppose that the following conditions hold:*
(i) *There exists an LS$[p](t,k',p^{s+1}+t)$ for every $t+1 \le k' \le min(k,(p^{s+1}+t)/2)$,*
(ii) *There exists an LS$[p](t,ip^n+j,p^{n+1}+t)$ for every i, j and n such that $0 \le j \le t, 1 \le i \le (p-1)/2, ip^n+j \le k$ and $n > s$.*
Then $A[p](t_1,k_1) = B[p](t_1,k_1)$ for all $p^s - 1 \le t_1 \le t$ and $t_1 < k_1 \le k$.

Theorem 5.6 **[18]** *Let $t \le 3^{f-2}$ and suppose that $A[3](t,i) = B[3](t,i)$ for $t < i < 3^f$. Let $3^{n-1} \le k < 3^n$ $(n > f)$. Then*
(i) *$B[3](t,k) \setminus A[3](t,k) \subset \{3^n + j \mid t \le j < t3^{n-f}\}$,*
(ii) *If $2 \cdot 3^{n-1} + t3^{n-f} \le k < 3^n$, then $A[3](t,k) = B[3](t,k)$.*

The following existence theorems on large sets of prime sizes which extend the known results are obtained by an examination of parameters of large sets which their existence was established in the previous sections.

Theorem 5.7 *If $k < 81$, then $A[3](3,k) = B[3](3,k)$.*

Proof The assertion holds for $k < 9$ by [19]. To fill in the whole range, by Theorem 5.5, we need to establish the existence of the following large sets:

(i) LS[3]$(3, 9, 30)$, (ii) LS[3]$(3, 10, 30)$, (iii) LS[3]$(3, 11, 30)$, (iv) LS[3]$(3, 12, 30)$,
(v) LS[3]$(3, 27, 84)$, (vi) LS[3]$(3, 28, 84)$, (vii) LS[3]$(3, 29, 84)$, (viii) LS[3]$(3, 30, 84)$.

Large set (i) exists by Table 3 and Theorem 5.1. Large sets (ii)-(iv) exist by Table 2. The existence of large sets (v)-(viii) was established in Section 2. $\square$

Theorem 5.8 *If $2 \cdot 3^{n-1} + 3^{n-3} \leq k < 3^n$ for some integer $n > 4$, then $A[3](3, k) = B[3](3, k)$.*

Proof This follows immediately from Theorems 5.6 and 5.7. $\square$

Theorem 5.9 *If $k \leq 10$, then $A[3](4, k) = B[3](4, k)$.*

Proof The assertion holds for $k < 9$ by [19]. Now by Theorem 5.5, we only need LS[3]$(4, 9, 31)$ and LS[3]$(4, 10, 31)$ which exist as Table 3 and Theorem 5.1 show. $\square$

Theorem 5.10 *If $k \leq 8$, then $A[5](2, k) = B[5](2, k) \setminus \{7\}$.*

Proof The assertion holds for $k \leq 5$ by [11, 15]. For higher k, first let $k = 6, 7$. By Theorem 5.2, we have

$$\mathrm{B}[5](2, k) = \{25l + i|\ l \geq 1, \quad i = 2, \ldots, k-1\}.$$

There exist LS[5]$(2, 6, 27)$ and LS[5]$(2, 7, 27)$ by Table 4 and Table 2, respectively and Theorem 5.1. Therefore, the assertion follows from Corollary 5.1. Now let $k = 8$. By Theorem 5.2, we have

$$\mathrm{B}[5](2, k) = \{5l + i|\ l \geq 1, \quad i = 2\} \cup \{25l + i|\ l \geq 1, \quad i = 2, \ldots, k-1\}.$$

There exists an LS[5]$(2, 8, 17)$ by Table 4. There also exist LS[5]$(2, 8, 22)$ and LS[5]$(2, 8, 27)$ by Table 4 and Table 2, respectively and Theorem 5.1. Hence we are done by Corollary 5.1. $\square$

Theorem 5.11 *If $k \leq 6$, then $A[5](3, k) = B[5](3, k) \setminus \{8\}$.*

Proof The assertion holds for $k \leq 5$ by [11, 15]. Let $k = 6$. By Theorem 5.2, we have

$$\mathrm{B}[5](3, k) = \{25l + i|\ l \geq 1, \quad i = 3, \ldots, k-1\}.$$

There is an LS[5]$(3, 6, 28)$ by Table 4 which is the only large set we need to make use of Corollary 5.1. $\square$

Theorem 5.12 *If $k < 7$, then $A[7](3,k) = B[7](3,k) \setminus \{10\}$.*

Proof By Theorem 5.2, we have

$$\mathrm{B}[7](3,k) = \{7l + i |\ l \geq 1, \quad i = 3, \ldots, k-1\}.$$

We use Theorem 5.3 and Corollary 5.1 to prove the statement. For $k = 4$, we need an $\mathrm{LS}[7](3,4,17)$ which exists by [7] and an $\mathrm{LS}[7](3,4,24)$ which exists by Table 4. For $k = 5$, we need $\mathrm{LS}[7](3,5,v)$ for $v = 10, 11, 17$ which exist by [12], [8] and Table 4, respectively. Finally, for $k = 6$, we need $\mathrm{LS}[7](3,6,12)$ which exists by [12] and $\mathrm{LS}[7](3,6,v)$ for $v = 17, 24$ which are shown as known in Table 4. □

Theorem 5.13 *If $k \leq 4$, then $A[p](2,k) = B[p](2,k)$ for $p = 13, 17, 19, 23$.*

Proof By Theorem 5.5, we need $\mathrm{LS}[p](2,k,p+2)$ for $k = 3, 4$ and $p = 13, 17, 19, 23$. These large sets exist for $k = 3$ by [16, 17]and for $k = 4$ by Table 4. □

Theorem 5.14 *If $k < 29$, then $A[29](2,k) = B[29](2,k)$.*

Proof By Theorem 5.5, we need $\mathrm{LS}[29](2,k,31)$ for $k \leq 15$. The large sets exist for $k = 3$ [16, 17], for $k = 4$ by [15], for $k = 5, 6$ by Table 4 and Theorem 5.1, and for $8 \leq k \leq 15$ by Table 2 and Theorem 5.1. Finally, there is an $\mathrm{LS}[29](2,7,31)$ by Table 4. □

We summarize the known results on the existence of large sets in Table 5.

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Table 5: Existence of large sets ($*$ means all admissible values)

N	t	k	v	Ref.
$*$	1	$*$	$*$	[4, 9]
$*$	2	3	$\neq 7$	[16, 17, 20]
2	2	$*$	$*$	[1]
2	≤ 5	< 16	$*$	[1, 3, 9, 14]
2	6	$7, 8, 9$	$*$	[1, 14]
3	2	< 81	$*$	[11]
3	3	< 81	$*$	Theorem 5.7
3	4	≤ 10	$*$	Theorem 5.9
5	2	≤ 8	$\neq 7$	Theorem 5.10
5	3	≤ 6	$\neq 8$	Theorem 5.11
7	2	< 7	$*$	[11, 15]
7	3	< 7	$\neq 10$	Theorem 5.12
11	2	≤ 10	$*$	[11, 15]
$13, 17, 19, 23$	2	$3, 4$	$*$	Theorem 5.13
29	2	< 29	$*$	Theorem 5.14

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