Yet another quick proof that $K_{10} \neq P + P + P$

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We give a new proof of the well known result that K_{10} has no decomposition into three disjoint copies of the Petersen graph P (for other proofs, see [1, 2, 3, 4, 5, 6]). We use the fact that in P any two adjacent (non-adjacent) vertices have exactly 0 (1) common neighbors. Suppose that $K_{10} = G^1 + G^2 + G^3$, where G^f $(f = 1, 2, 3)$ are copies of P. Consider a partition of the vertices of K_{10} into $V = \{1, 2, \ldots, c\}$ and $W = \{c+1, \ldots, 10\}$. Let k_i^f and $3 - l_j^f$ be the degrees of the vertices i $(1 \leq i \leq c)$ and j $(c+1 \leq j \leq 10)$ in the induced subgraphs of G^f on V and W, respectively. Then with two simple double counting and using the fact mentioned above we have

$$\sum_{i=1}^{c} k_i^f + \sum_{j=c+1}^{10} l_j^f = 3c,$$

$$\sum_{i=1}^{c} (k_i^f)^2 + \sum_{j=c+1}^{10} l_j^f (l_j^f - 1) = c(c-1),$$

for $f = 1, 2, 3$. For our purpose, we take $c = 5$ and V such that $k_i^1 = 2$ and $l_j^1 = 1$. So $(k_i^2, k_i^3) \in \{(0,2), (2,0), (1,1)\}$ and $(l_j^2, l_j^3) \in \{(1,3), (3,1), (2,2)\}$. Let x_1, x_2, x_3 be the numbers of (k_i^2, k_i^3) which are $(0,2), (2,0), (1,1)$, respectively and let y_1, y_2, y_3 be the numbers of (l_j^2, l_j^3) which are $(1,3), (3,1)$,

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$(2,2)$, respectively. Then the above equations are reduced to

$$\begin{aligned}
2x_2 + x_3 + y_1 + 3y_2 + 2y_3 &= 15,\\
4x_2 + x_3 + 6y_2 + 2y_3 &= 20,\\
2x_1 + x_3 + 3y_1 + y_2 + 2y_3 &= 15,\\
4x_1 + x_3 + 6y_1 + 2y_3 &= 20.
\end{aligned}$$

Adding up these equations with coefficients $6, -3, -4, 1$, respectively results in $x_1 + y_2 = 5/2$ which is a contradiction.

References

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