# On large sets of $t$-designs of size four[1]

G. B. Khosrovshahi[a,b,2] R. Laue[c] B. Tayfeh-Rezaie[a,3]

[a] *Institute for Studies in Theoretical Physics and Mathematics (IPM), P.O. Box* 19395-5746*, Tehran, Iran*
[b] *Department of Mathematics, University of Tehran, Tehran, Iran*
[c] *Mathematical Department, University of Bayreuth, D*-95440 *Bayreuth, Germany*

**Abstract**

A set of well known necessary conditions for the existence of a large set of $t$-designs, LS$[N](t,k,v)$, is $N \Big| \binom{v-i}{k-i}$ for $i = 0,\ldots,t$. We investigate the existence of large sets of size four. We take advantages of the recursive and direct constructions to show that the trivial necessary conditions are sufficient when $N = 4, t = 2, 3$ and $k \leq 7$.

**Keywords:** $t$-designs, large sets of $t$-designs, partitionable sets, recursive constructions, direct constructions
**MR Subject Classification:** 05B05

## 1 Introduction

Let $t, k, v$ and $\lambda$ be integers such that $0 < t \leq k \leq v$ and $\lambda > 0$. Let $X$ be a $v$-set and $P_k(X)$ denote the set of all $k$-subsets of $X$. A $t$-$(v, k, \lambda)$ *design* is a pair $(X, \mathcal{D})$ in which $\mathcal{D}$ is a collection of elements

[1] This research was partially supported by a grant from IPM.
[2] Corresponding author, email: rezagbk@ipm.ir.
[3] Supported by a grant from IPM (No. 83050312).

of $P_k(X)$ (called *blocks*) such that every $t$-subset of $X$ appears in exactly $\lambda$ blocks. Let $N > 1$. A *large set* of $t$-$(v, k, \lambda)$ designs of size $N$, denoted by LS$[N](t, k, v)$, is a set of $N$ disjoint $t$-$(v, k, \lambda)$ designs $(X, \mathcal{D}_i)$ such that $\{\mathcal{D}_i | \ 1 \leq i \leq N\}$ is a partition of $P_k(X)$. Note that we have $N = \binom{v-t}{k-t}/\lambda$. A set of well known necessary conditions for the existence of an LS$[N](t, k, v)$ is

$$N \Big| \binom{v-i}{k-i}, \qquad 0 \leq i \leq t. \tag{1.1}$$

The central question on large sets is the existence problem. This question has been completely answered for large sets of 1-designs, of triple systems, of 2-designs of size 2. Apart from these comprehensive results, there are also some partial results. For a review of known results we refer the reader to [10].
The methods of constructions of large sets can be divided into two categories: Direct and recursive constructions. The large sets found via the former methods are utilized as initial structures in the latter methods and usually a combined usage of direct and recursive constructions can lead to the general existence results. This approach has been successfully used in establishing many existence results for large sets of $t$-designs of prime sizes (see [2, 10, 11, 14, 16]). In this paper, we investigate the existence of large sets of size four. We show that the trivial necessary conditions (1.1) are sufficient for $N = 4, t = 2, 3$ and $k \leq 7$.

## 2 Recursive constructions

Many recursive constructions of large sets are obtained via the notion of $(N, t)$-partitionable sets which was first introduced in [3]. This idea is indeed a generalization of the notion of large sets, where we consider $t$-balanced partition of a subset $\mathcal{B}$ of $P_k(X)$ instead of the whole set $P_k(X)$. Let $\mathcal{B}_1, \mathcal{B}_2 \subseteq P_k(X)$. We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are *$t$-equivalent* if every $t$-subset of $X$ appears in the same number of blocks of $\mathcal{B}_1$ and $\mathcal{B}_2$. If there exists a partition of $\mathcal{B} \subseteq P_k(X)$ into $N$ mutually

$t$-equivalent subsets, then $\mathcal{B}$ is called an $(N,t)$-*partitionable* set. Let $X_1$ and $X_2$ be two disjoint sets and let $\mathcal{B}_i \subseteq P_{k_i}(X_i)$ for $i = 1, 2$. Then we define

$$\mathcal{B}_1 * \mathcal{B}_2 = \{B_1 \cup B_2 | \ B_1 \in \mathcal{B}_1, \ B_2 \in \mathcal{B}_2\}.$$

The following two lemmas concern $(N,t)$-partitionable sets. The first one is rather trivial but the second one contains an unexpected result.

**Lemma 2.1** **[3]** (i) *$t$-equivalence implies $i$-equivalence for all* $0 \leq i \leq t$.
(ii) *The union of disjoint $(N,t)$-partitionable sets is again an $(N,t)$-partitionable set.*

**Lemma 2.2** **[3]** *Let $X_1$ and $X_2$ be two disjoint sets and let $\mathcal{B}_i \subseteq P_{k_i}(X_i)$ for $i = 1, 2$. Suppose that $\mathcal{B}_1$ is $(N,t_1)$-partitionable. Then*

(i) *$\mathcal{B}_1 * \mathcal{B}_2$ is $(N,t_1)$-partitionable.*
(ii) *If $\mathcal{B}_2$ is $(N,t_2)$-partitionable, then $\mathcal{B}_1 * \mathcal{B}_2$ is $(N,t_1+t_2+1)$-partitionable.*

In order to find an LS$[N](t,k,v)$ on a $v$-set $X$, we try to partition $P_k(X)$ in such a way that each part of the partition is an $(N,t)$-partitionable set. If this is done, then by Lemma 2.1, $P_k(X)$ will be an $(N,t)$-partitionable set which means that we have obtained an LS$[N](t,k,v)$. This general approach has been used to obtain great numbers of recursive constructions. We illustrate the method by the following two examples.

**Theorem 2.1** **[3]** *If there exist LS$[N](t,k,v)$ and LS$[N](t,k+1,v)$, then there exists an LS$[N](t,k+1,v+1)$.*

**Proof** Let $X$ be a $v$-set and $x \notin X$. Consider the following partitioning of $P_{k+1}(X \cup \{x\})$:

$$\mathcal{B}_0 = P_{k+1}(X),$$
$$\mathcal{B}_1 = \{\{x\}\} * P_k(X).$$

By the assumption $\mathcal{B}_0$ is $(N,t)$-partitionable. Also $P_k(X)$ is an $(N,t)$-partitionable set by the assumption and therefore by Lemma 2.2, $\mathcal{B}_1$ is $(N,t)$-partitionable. Now the assertion follows from Lemma 2.1. □

**Theorem 2.2** **[3]** *If LS*$[N](t,i,v)$ *exist for all* $t+1 \le i \le k$ *and an LS*$[N](t,k,u)$ *also exists, then an LS*$[N](t,k,u+v-t)$ *exists.*

**Proof** Let $X = \{1,2,\ldots,u+v-t\}$ and also for $1 \le j \le u+v-t$, let $X_j = \{1,2,\ldots,j\}$ and $Y_j = X \setminus X_j$. For $0 \le i \le k$, define

$$\mathcal{B}_i = P_{k-i}(X_{u-i}) * P_i(Y_{u-i+1}).$$

Then it can be shown that $\mathcal{B}_i$ provide a partitioning of $P_k(X)$.
By Lemma 2.1, it suffices to show that each $\mathcal{B}_i$ is $(N,t)$-partitionable. Let $1 \le i \le t$. Since there exists an LS$[N](t,k,u)$, hence there exists an LS$[N](t-i,k-i,u-i)$. This means that $P_{k-i}(X_{u-i})$ is $(N,t-i)$-partitionable. On the other hand from LS$[N](t,t+1,v)$ we find LS$[N](i-1,i,v-t-1+i)$. Therefore $P_i(Y_{u-i+1})$ is $(N,i-1)$-partitionable and so from Lemma 2.2 it turns out that $\mathcal{B}_i$ is an $(N,t)$-partitionable set. Now let $t+1 \le i \le k$. From the assumption and Theorem 2.1, it is seen that there exists an LS$[N](t,i,v-t-1+i)$. Hence $P_i(Y_{u-i+1})$ is $(N,t)$-partitionable and so is $\mathcal{B}_i$. Finally, we note that $\mathcal{B}_0$ is $(N,t)$-partitionable by the assumption. □

The following corollary is a useful application of Theorems 2.1 and 2.2. We will use this recursive theorem later on to obtain new results on large sets of size 4.

**Corollary 2.1** *If LS*$[N](t,i,v)$ *exist for all* $t+1 \le i \le k$*, then LS*$[N](t,i,l(v-t)+j)$ *exist for all* $l \ge 1$*,* $t+1 \le i \le k$ *and* $t \le j < i$*.*

## 3 Direct constructions

The most common direct method for constructing $t$-designs and large sets is based on group actions. In this method one tries to find designs using prescribed automorphism groups. The main point is that the block sets of such designs must be a union of some orbits of the prescribed group on all those subsets of the point set which are of the same size. This method was formulated by Kramer and Mesner in [12] as a matrix equation. Formally, let $G$ be a permutation group on a set $X$ of size $v$ and let $t$ and $k$ be integers such that $0 < t \leq k \leq v$. Let $T_1, T_2, \ldots, T_s$ and $K_1, K_2, \ldots, K_r$ be the orbits under the induced action of $G$ on $t$-subsets and $k$-subsets of $X$, respectively. Denote by $A_{tk}^v(G) = (a_{ij})$ the $s \times r$ matrix, where $a_{ij}$ is the number of $k$-subsets in the orbit $K_j$ containing a representative $t$-subset in the orbit of $T_i$. By [12], this matrix is well defined and a $t$-$(v, k, \lambda)$ design with automorphism group $G$ exists if and only if there is a $(0, 1)$-vector $u$ satisfying the equation $A_{tk}^v(G)u = \lambda J$, where $J$ is the all one vector. Clearly, one may also adapt the same method for finding large sets (see [14]). We pick up one from the set of solutions of $u$ and remove the corresponding columns from $A_{tk}^v(G)$. The resulting matrix $A_{tk}'^v(G)$ is used in a similar way to $A_{tk}^v(G)$ to find designs via the equation $A_{tk}'^v(G)u' = \lambda J$. We repeat the procedure until all orbits on $k$-subsets are used or no design is obtained in some step. In the former case we have a large set and in the latter we should backtrack and try another solution from an earlier step. For practical purposes this algorithm can be modified in order to search for large sets in a randomized manner.

Many designs and large sets have been found by this approach computationally and theoretically (for theoretical results, see for example [4, 5, 7, 8, 9, 14, 15]). A great amount of work has been done to solve the matrix equation via computational algorithms. The most natural method to solve such systems is a backtracking algorithm widely used by many authors. There is a different approach which uses lattice base reduction by the LLL-algorithm to obtain a basis for the set of solutions which contains short vectors and then applies a backtracking algorithm to find $(0, 1)$ solutions. The use of LLL-algorithm for finding $t$-designs was first introduced by Kreher and Radziszowski in [13] and the above sketchy presentation of an improved version is due to Wassermann [17]. This method is the basis of a computer program named DISCRETA to construct $t$-designs with prescribed automorphism groups developed by Betten, Haberberger, Laue, Wassermann at the University of Bayreuth. The program has the ability of random search for large sets. We use this program to find some new large sets in the subsequent sections.

# 4 Large sets of size four

In this section, we take advantage of the direct and recursive constructions to obtain some new results on large sets of size 4. The following theorem is useful for identifying feasible parameters. Let $m$ and $n$ be positive integers. We denote the quotient and the remainder of division $m$ by $n$ by $[m/n]$ and $(m/n)$, respectively. Let $N, t$ and $k$ be given. The set of all $v$ which satisfy the necessary conditions (1.1) is denoted by $\mathrm{B}[N](t, k)$.

**Theorem 4.1** **[11]** *Let $p^\alpha$ be a prime power. $v \in B[p^\alpha](t, k)$ if and only if there exist distinct positive integers $\ell_i$ $(1 \le i \le \alpha)$ such that $t \le (v/p^{\ell_i}) < (k/p^{\ell_i})$.*

**Lemma 4.1** *Let $v > 7$. Then*
(i) *$v \in B[4](2, 3)$ if and only if $v \equiv 2 \pmod 8$.*
(ii) *$v \in B[4](2, 4)$ if and only if $v \equiv 2, 3 \pmod{16}$.*
(iii) *$v \in B[4](2, 5)$ if and only if $v \equiv 2, 3, 4 \pmod{16}$.*
(iv) *$v \in B[4](2, 6)$ if and only if $v \equiv 2, 3, 4, 5 \pmod{16}$.*
(v) *$v \in B[4](2, 7)$ if and only if $v \equiv 2, 3, 4, 5, 6, 10, 14 \pmod{16}$.*

**Proof** We prove (v). The proofs of the other cases are similar. Let $v \in \mathrm{B}[4](2, 7)$. Then by Theorem 4.1, there are distinct integers $0 < z_1 < z_2$ such that $2 \le (v/2^{z_i}) < (7/2^{z_i})$ for $i = 1, 2$. It is clear that $z_1 \ge 2$. First let $z_1 = 2$. Then $v = 4v_1 + 2 = v_3 2^{z_2} + j$, where $j = 2, 6$ and so $v \equiv 2, 6, 10, 14 \pmod{16}$. Now suppose that $z_1 > 2$. Then $v = v_1 2^{z_2} + j$, where $j = 2, 3, 4, 5, 6$ and so $v \equiv 2, 3, 4, 5, 6 \pmod{16}$. The converse is easily checked using Theorem 4.1. □

**Lemma 4.2** *Let $4 \leq k \leq 7$ and $v > k$. Then $v \in B[4](3,k)$ if and only if $v \equiv 3, 4, \ldots, k-1 \pmod{16}$.*

**Proof** The proof is similar to that of Lemma 4.1. □

**Theorem 4.2** *Let $t = 2, 3$ and $t < k \leq 7$. Then there exists an LS[4]$(t,k,v)$ if and only if the necessary conditions* (1.1) *hold.*

**Proof** By Corollary 2.1 and Lemmas 4.1 and 4.2, it suffices to establish the existence of the following large sets: LS[4]$(2,i,18)$ and LS[4]$(3,i,19)$ for $4 \leq i \leq 7$, LS[4]$(2,3,10)$, LS[4]$(2,7,10)$ and LS[4]$(2,7,14)$. Regarding derived and complementary designs, we only need to find LS[4]$(3,i,19)$ for $4 \leq i \leq 7$, LS[4]$(2,3,10)$ and LS[4]$(2,7,14)$. These large sets are constructed in the next section. □

We conjecture that for all feasible values of $k$ and $v$, LS[4]$(2,k,v)$ and LS[4]$(3,k,v)$ exist. To settle this conjecture, one should first determine the parameters of the so-called root cases, i. e. large sets which can not be obtained via the known recursive constructions. We think that this part can be resolved through Theorem 4.1. The next part is to find a way of construction of root cases. This part needs a theoretical approach since the computer search is obviously applicable for only a few sets of small parameters. One possible approach may be similar to that one used by Ajoodani in [1] to construct root cases of large sets of 2-designs of size 2 through partitionable sets. We also note that Ajoodani's construction is very complicated and it will be a hard task to extend it to large sets of size 4.

# 5 Root cases

The recursive constructions are based on the following large sets that are constructed using DISCRETA. We need large sets LS[4]$(3,i,19)$ for $4 \leq i \leq 7$, LS[4]$(2,3,10)$ and LS[4]$(2,7,14)$. Note that an LS[4]$(2,3,10)$ is known [6]. For LS[4]$(3,i,19)$ $(4 \leq i \leq 7)$, we found

that the prescribed group $D_{17}++$, that is the dihedral group on 17 points with two fixed points added, acts as a group of automorphisms of each of 4 disjoint designs.
In the case of LS[4](3, 4, 19), a slightly larger group could be taken: The semidirect product $C_{17} \rtimes C_4$, of $C_{17}$ with its group of automorphisms of order 4 in its natural action on $C_{17}$. The Kramer-Mesner matrix in this case is of size $21 \times 72$. The disjoint designs were found by iteratively selecting at random one solution of a 3-design with the parameters needed and deleting the corresponding columns from the Kramer-Mesner matrix. This procedure is iterated until a large set is found or there is no solution. In general, several runs are needed to find a large set. Since the random number generator is based on tables, the solutions can be reproduced by starting the program with identical input again.
The group $C_{17} \rtimes C_4$ of order 68 is given by the following generators:

$$(0)(1)(2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18),$$

$$(0)(1)(2)(3,15,18,6)(4,11,17,10)(5,7,16,14)(8,12,13,9).$$

The corresponding orbits (in order similar to the order of columns of the Kramer-Mesner matrix) are represented by the following sets. In each case the subscript gives the size of the stabilizer of the set.

$$\{0,1,2,3\}_2, \{0,1,2,4\}_2, \{0,1,2,5\}_2, \{0,1,2,8\}_2, \{0,2,3,4\}_2,$$

$$\{0,2,3,5\}_1, \{0,2,3,6\}_1, \{0,2,3,8\}_1, \{0,2,3,9\}_1, \{0,2,3,10\}_1,$$

$$\{0,2,3,11\}_2, \{0,2,4,7\}_1, \{0,2,4,9\}_1, \{0,2,4,10\}_1, \{0,2,5,8\}_2,$$

$$\{0,2,5,12\}_2, \{1,2,3,4\}_2, \{1,2,3,5\}_1, \{1,2,3,6\}_1, \{1,2,3,8\}_1,$$

$$\{1,2,3,9\}_1, \{1,2,3,10\}_1, \{1,2,3,11\}_2, \{1,2,4,7\}_1, \{1,2,4,9\}_1,$$

$$\{1,2,4,10\}_1, \{1,2,5,8\}_2, \{1,2,5,12\}_2, \{2,3,4,5\}_2, \{2,3,4,6\}_1,$$

$$\{2,3,4,7\}_1, \{2,3,4,8\}_1, \{2,3,4,9\}_1, \{2,3,4,10\}_1, \{2,3,4,11\}_1,$$

$$\{2,3,5,6\}_2, \{2,3,5,7\}_1, \{2,3,5,8\}_1, \{2,3,5,9\}_1, \{2,3,5,10\}_1$$

$$\{2,3,5,11\}_1, \{2,3,5,12\}_1, \{2,3,5,13\}_1, \{2,3,5,14\}_1, \{2,3,5,15\}_1,$$
$$\{2,3,5,16\}_1, \{2,3,5,17\}_2, \{2,3,6,7\}_4, \{2,3,6,8\}_1, \{2,3,6,9\}_1,$$
$$\{2,3,6,12\}_1, \{2,3,6,13\}_1, \{2,3,6,16\}_2, \{2,3,8,9\}_2, \{2,3,8,10\}_1,$$
$$\{2,3,8,11\}_1, \{2,3,8,13\}_1, \{2,3,8,14\}_2, \{2,3,9,10\}_2, \{2,3,9,11\}_1,$$
$$\{2,3,9,12\}_1, \{2,3,9,13\}_4, \{2,3,10,11\}_2, \{2,3,10,12\}_2, \{2,4,7,9\}_2,$$
$$\{2,4,7,10\}_1, \{2,4,7,13\}_1, \{2,4,7,14\}_1, \{2,4,7,16\}_4, \{2,4,9,11\}_2,$$
$$\{2,4,9,14\}_2, \{2,4,10,12\}_4.$$

The Kramer-Mesner matrix $A^{19}_{3,4}$ is as follows.

```
444400000000000000000000000000000000000000000000000000000000000000000000
100022422210000000000000000000000000000000000000000000000000000000000000
010012000222240000000000000000000000000000000000000000000000000000000000
001002220004202100000000000000000000000000000000000000000000000000000000
000100024200221200000000000000000000000000000000000000000000000000000000
100000000000000022422210000000000000000000000000000000000000000000000000
010000000000000012000222240000000000000000000000000000000000000000000000
001000000000000000222000420210000000000000000000000000000000000000000000
000100000000000000024200221200000000000000000000000000000000000000000000
000010000000000010000000000022222220000000000000000000000000000000000000
000001000000000001000000000011000001111111111110000000000000000000000000
000000100000000000100000000001210001100000001101111110000000000000000000
000000010000000000010000000000000110000110000110001100111111000000000000
000000001000000000001000000000110000100010000001120100101111000000000000
000000000100000000000100000000010110001101000100010001000101011000000000
000000000010000000000010000002000020200020000000000000200020020000000000
000000000001000000000001000000100000110100010010100000001000000012111000
000000000000100000000000100000000010000001101010000010001100001000101101
000000000000010000000000010000000001100000111100000000001000020001011001
000000000000001000000000001000000000000000200000020002000000200000000020
000000000000000100000000000000000000000000200000000000000000000000000000
```

The solution vectors for an LS[4](3,4,19) are as follows.

```
1000000010110000000010010100000001010100001000011001000000011000000000101100
0010001000000101010001000001100000100001000010000000111010000001010100010
0001010001000010001000000110010010000001000010001100000000001100010010000
0100100100001000100010010000000100000110001100000000100010100000100000001
```

In the cases $k = 5, 6, 7$ and $v = 19$ with the group $D_{17}++$, the Kramer Mesner matrices are of size:

- $41 \times 388$ for $k = 5$,
- $41 \times 868$ for $k = 6$,
- $41 \times 1580$ for $k = 7$.

For LS[4]$(2, 7, 14)$ we found a solution using the group $C_{13} \rtimes C_4$. The matrices and the solutions are available from the authors. It should be noted that a number of other groups like $C_{19}$ did not lead to the desired large sets.

**Acknowledgement** This research was carried out while the second author was visiting the IPM. This author greatly acknowledges the hospitality and fruitful working atmosphere of the institute.

## References

[1] S. Ajoodani-Namini, *All block designs with* $b = \binom{v}{k}/2$ *exist*, Discrete Math. **179** (1998), 27–35.

[2] S. Ajoodani-Namini, *Extending large sets of t-designs*, J. Combin. Theory Ser. A **76** (1996), 139–144.

[3] S. Ajoodani-Namini and G. B. Khosrovshahi, *More on halving the complete designs*, Discrete Math. **135** (1994), 29–37.

[4] P. J. Cameron, H. R. Maimani, G. R. Omidi and B. Tayfeh-Rezaie, 3-*Designs from* $PSL(2, q)$, submitted.

[5] P. J. Cameron, G. R. Omidi and B. Tayfeh-Rezaie, 3-*Designs from* $PGL(2, q)$, preprint.

[6] Y. M. Chee and S. S. Magliveras, *A few more large sets of t-designs*, J. Combin. Des. **6** (1998), 293–308.

[7] C. A. Cusack, S. W. Graham and D. L. Kreher, *Large sets of* 3*-designs from* $PSL(2, q)$ *with block sizes* 4 *and* 5, J. Combin. Des. **3** (1995), 147–160.

[8] C. A. Cusack and S. S. Magliveras, *Semiregular large sets of $t$-designs*, Des. Codes Cryptogr. **18** (1999), 81–87.

[9] S. Iwasaki, *Infinite families of* 2- *and* 3*-designs with parameters* $v = p + 1$, $k = (p - 1)/2^i + 1$, *where* $p$ *odd prime,* $2^e \top (p - 1)$, $e \geq 2$, $1 \leq i \leq e$, J. Combin. Des. **5** (1997), 95–110.

[10] G. B. Khosrovshahi and B. Tayfeh-Rezaie, *Large sets of $t$-designs through partitionable sets: A survey,* Discrete Math., to appear.

[11] G. B. Khosrovshahi and B. Tayfeh-Rezaie, *Root cases of large sets of $t$-designs,* Discrete Math. **263** (2003), 143–155.

[12] E. S. Kramer and D. M. Mesner, *$t$-designs on hypergraphs*, Discrete Math. 15 (1976), 263–296.

[13] D. L. Kreher and S. P. Radziszowski, *The existence of simple* 6-(14, 7, 4) *designs*, J. Combin. Theory Ser. A **43** (1986), 237–243.

[14] R. Laue, S. S. Magliveras and A. Wassermann, *New large sets of $t$-designs*, J. Combin. Des. **9** (2001), 40–59.

[15] G. R. Omidi, M. R. Pournaki and B. Tayfeh-Rezaie, 3-*Designs from $PSL(2, q)$ with block size* 6 *and their large sets*, submitted.

[16] B. Tayfeh-Rezaie, *On the existence of large sets of $t$-designs of prime sizes*, Des. Codes Cryptogr., to appear.

[17] A. Wassermann, *Finding simple $t$-designs with enumeration techniques,* J. Combin. Des. **6** (1998), 79–90.