Root cases of large sets of t-designs*

G. B. Khosrovshahi[a,b,†] and B. Tayfeh-Rezaie[a]

[a]*Institute for Studies in Theoretical Physics and Mathematics (IPM)*
P.O. Box 19395-5746*, Tehran, Iran*
[b]*Department of Mathematics, University of Tehran, Tehran, Iran*

Abstract

A large set of t-(v,k,λ) designs of size N, denoted by $\mathrm{LS}[N](t,k,v)$, is a partition of all k-subsets of a v-set into N disjoint t-(v,k,λ) designs, where $N = \binom{v-t}{k-t}/\lambda$. A set of trivial necessary conditions for the existence of an $\mathrm{LS}[N](t,k,v)$ is $N \big| \binom{v-i}{k-i}$ for $i = 0,\ldots,t$. In this paper we extend some of the recursive methods for constructing large sets of t-designs of prime sizes. By utilizing these methods we show that for the construction of all possible large sets with the given N, t, and k, it suffices to construct a finite number of large sets which we call *root cases.* As a result, we show that the trivial necessary conditions for the existence of $\mathrm{LS}[3](2,k,v)$ are sufficient for $k \le 80$.

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†Corresponding author, IPM, P.O. Box 19395-5746, Tehran, Iran. Email: rezagbk@ipm.ir.

1. Introduction

A t-(v,k,λ) design is a collection of k-subsets of a given v-set such that every t-subset of the v-set is exactly contained in λ elements of the collection. A large set of t-(v,k,λ) designs of size N, denoted by $\mathrm{LS}[N](t,k,v)$, is a partition of all k-subsets of a given v-set into N disjoint t-(v,k,λ) designs, where $N = \binom{v-t}{k-t}/\lambda$. A set of necessary conditions for the existence of an $\mathrm{LS}[N](t,k,v)$ is $N \big| \binom{v-i}{k-i}$ for $i = 0,\ldots,t$. In 1987, A. Hartman [9] conjectured that these necessary conditions are sufficient for the existence of large sets of size $N = 2$. Then, the first author proposed similar conjectures for $N = 3, 4$ [3]. These conjectures have not yet been settled and their proofs seem to be far from reach. S. Ajoodani-Namini established the truth of Hartman's conjecture for $t = 2$ [1]. For $t > 2$, there exist some partial results. For $N = 3$, the problem has been solved for $t \leq 4$ and $k \leq 8$ [19].

Along this line of thinking, some recursive constructions, with some merits have been introduced. They have been instrumental in the production of many infinite families of large sets. Most of those recursive constructions are based on the notion of (N,t)-partitionable sets which was initiated in [4]. This notion is in fact a generalization of large sets. Utilizing these recursive constructions, one can reduce the proof of Hartman's conjecture to the question of existence of certain classes of large sets which we call *root cases*.

In this paper, we develop some recursive constructions based on the notion of (N,t)-partitionable sets for large sets of prime sizes. This allows us to determine the root cases for large sets of prime sizes. Consequently, we show that the necessary conditions for the existence of $\mathrm{LS}[3](2,k,v)$ are sufficient for $k \leq 80$.

2. Definitions and Preliminaries

Let t, k, v and λ be integers such that $0 \le t \le k \le v$ and $\lambda \ge 1$ and let X be a v-set. We denote the set of all k-subsets of X by $P_k(X)$. A t-(v, k, λ) *design* (briefly a t-design) on X is a collection $\mathcal{D}$ of k-subsets of X (called *blocks*) such that every t-subset of X is contained in exactly λ blocks of $\mathcal{D}$. Hereafter we assume that $0 \le t < k < v$ to avoid trivial cases. A t-design with no repeated block is called *simple* t-design. Here we are only concerned with simple t-designs. $P_k(X)$ is trivially a t-$(v, k, \binom{v-t}{k-t})$ design which is called the *complete* design. A simple counting argument shows that a t-(v, k, λ) design is also an i-(v, k, λ_i) design, for $0 \le i \le t$, where $\lambda_i = \lambda\binom{v-i}{t-i}/\binom{k-i}{t-i}$. Hence, a set of necessary conditions for the existence of a t-(v, k, λ) design is

$$\lambda\binom{v-i}{t-i} \equiv 0 \pmod{\binom{k-i}{t-i}}, \qquad 0 \le i \le t. \tag{1}$$

Using $\binom{v-i}{t-i}\binom{v-t}{k-t} = \binom{v-i}{k-i}\binom{k-i}{t-i}$, the conditions (1) are equivalent to

$$\lambda\binom{v-i}{k-i} \equiv 0 \pmod{\binom{v-t}{k-t}}, \qquad 0 \le i \le t. \tag{2}$$

The least value of λ satisfying in (1) is denoted by $\lambda_{\min}$ and any other feasible λ is clearly an integer multiple of $\lambda_{\min}$. The λ of the complete design is denoted by $\lambda_{\max}$.

Let $\mathcal{D}$ be a t-(v, k, λ) design on X and let $x \in X$. We define

$$\begin{aligned} \mathcal{D}_d(x) &= \{B \setminus \{x\} \mid x \in B \in \mathcal{D}\}, \\ \mathcal{D}_r(x) &= \{B \mid x \notin B \in \mathcal{D}\}, \\ \mathcal{D}_c &= \{X \setminus B \mid B \in \mathcal{D}\}. \end{aligned}$$

One can easily see that $\mathcal{D}_d(x)$ and $\mathcal{D}_r(x)$ are $(t-1)$-$(v-1, k-1, \lambda)$ and $(t-1)$-$(v-1, k, \lambda_{t-1} - \lambda)$ designs, respectively, and are called *derived* and *residual* designs of $\mathcal{D}$ with respect to x. By the inclusion-exclusion principle, it is also seen that for $k \le v - t$, $\mathcal{D}_c$ is a t-$(v, v-k, \lambda_c)$ design, where $\lambda_c = \sum_{i=0}^{t}(-1)^t\binom{t}{i}\lambda_i$ and is called the *complement* of $\mathcal{D}$.

A *large set* of t-(v,k,λ) designs on X, denoted by $\mathrm{LS}_\lambda(t,k,v)$ or $\mathrm{LS}[N](t,k,v)$, is a partition $\mathcal{L}$ of $P_k(X)$ into N disjoint t-(v,k,λ) designs $\mathcal{D}^i$, where $N=\binom{v-t}{k-t}/\lambda$. By convention, we always assume that $N>1$. By (2), we observe that a set of necessary conditions for the existence of an $\mathrm{LS}[N](t,k,v)$ is

$$\binom{v-i}{k-i} \equiv 0 \pmod N, \qquad 0 \le i \le t. \tag{3}$$

The derived, residual, and complement large sets of $\mathcal{L}=\{\mathcal{D}^i\}$ are defined as $\mathcal{L}_d(x)=\{\mathcal{D}^i_d(x)\}$, $\mathcal{L}_r(x)=\{\mathcal{D}^i_r(x)\}$ and $\mathcal{L}_c=\{\mathcal{D}^i_c\}$ which are $\mathrm{LS}[N](t-1,k-1,v-1)$, $\mathrm{LS}[N](t-1,k,v-1)$ and $\mathrm{LS}[N](t,v-k,v)$ large sets, respectively. Note that we can obtain more large sets from a given large set which is shown in the following modified form of a theorem in [2].

Theorem 1. If there exists an $\mathrm{LS}[N](t,k,v)$, then there exist $\mathrm{LS}[N](t-i,k-j,v-l)$ for all $0\le j\le l\le i\le t$.

Proof. We prove the statement by induction on t. From the derived and residual large sets $\mathrm{LS}[N](t-1,k-1,v-1)$ and $\mathrm{LS}[N](t-1,k,v-1)$ and by the induction hypothesis we obtain $\mathrm{LS}[N](t-i,k-j,v-l)$ for $l\ge 1$ and $0\le j\le l\le i\le t$. On the other hand $\mathrm{LS}[N](t,k,v)$ is at the same time $\mathrm{LS}[N](i,k,v)$ for $0\le i\le t$. This completes the proof. $\square$

The following well known and simple extension theorem yields immediately Theorem 3 which will be useful in our work.

Theorem 2 [4]. If there exist $\mathrm{LS}[N](t,k,v)$ and $\mathrm{LS}[N](t,k+1,v)$, then there exists $\mathrm{LS}[N](t,k+1,v+1)$.

Theorem 3. If there exist $\mathrm{LS}[N](t,k+i,v)$ for all $0\le i\le l$, then there exist $\mathrm{LS}[N](t,k+i,v+j)$ for all $0\le j\le i\le l$.

Another useful extension theorem is the following theorem due to Alltop.

Theorem 4 [5]. Let t be even and N be arbitrary or let t be odd and $N = 2$. If there exists $\mathrm{LS}[N](t, k, 2k+1)$, then there exists $\mathrm{LS}[N](t+1, k+1, 2k+2)$.

Notation. Let N, t, and k be given. The set of all v's for which $\mathrm{LS}[N](t, k, v)$ exist is denoted by $\mathrm{A}[N](t, k)$. The set of all v's which satisfy the necessary conditions (3) is denoted by $\mathrm{B}[N](t, k)$.

3. (N, t)-Partitionable Sets

In this section we review the notion of (N, t)-partitionable sets which was introduced in [4]. This idea is indeed a generalization of the notion of large sets, where we consider a t-balanced partition of a subset $\mathcal{B}$ of $P_k(X)$ instead of the complete $P_k(X)$. More precisely, let $\mathcal{B}_1, \mathcal{B}_2 \subseteq P_k(X)$. We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are *t-equivalent* if every t-subset of X appears in the same number of blocks of $\mathcal{B}_1$ and $\mathcal{B}_2$. If there exists a partition of $\mathcal{B} \subseteq P_k(X)$ into N mutually t-equivalent subsets, then $\mathcal{B}$ is called an *(N, t)-partitionable* set. Let X_1 and X_2 be two disjoint sets and let $\mathcal{B}_i \subseteq P_{k_i}(X_i)$ for $i = 1, 2$. Then we define

$$\mathcal{B}_1 * \mathcal{B}_2 = \{B_1 \cup B_2 |\ B_1 \in \mathcal{B}_1,\ B_2 \in \mathcal{B}_2\}.$$

Lemma 1 [4]. (i) t-equivalence implies i-equivalence for all $0 \le i \le t$.
(ii) The union of disjoint (N, t)-partitionable sets is again an (N, t)-partitionable set.

Lemma 2 [4]. Let X_1 and X_2 be two disjoint sets and let $\mathcal{B}_i \subseteq P_{k_i}(X_i)$ for $i = 1, 2$. Suppose that $\mathcal{B}_1$ is (N, t_1)-partitionable. Then

(i) $\mathcal{B}_1 * \mathcal{B}_2$ is (N, t_1)-partitionable.

(ii) If $\mathcal{B}_2$ is (N, t_2)-partitionable, then $\mathcal{B}_1 * \mathcal{B}_2$ is (N, t_1+t_2+1)-partitionable.

By Lemma 1(ii), if we are able to partition $P_k(X)$ into disjoint (N, t)-partitionable sets, then we obtain a large set. This technique in combination with Lemma 2 provides a general approach for recursive and direct constructions of large sets. We first outline the approach by a simple example.

Example. Construction of an LS[2]$(2, 3, 10)$ from an LS[2]$(2, 3, 6)$. Let X be a 10-set and consider the following partition of $P_3(X)$:

$$\begin{aligned}
\mathcal{B}_1 &= P_3(\{1, \dots, 6\}), \\
\mathcal{B}_2 &= P_2(\{1, \dots, 5\}) * P_1(\{7, \dots, 10\}), \\
\mathcal{B}_3 &= P_1(\{1, \dots, 4\}) * P_2(\{6, \dots, 10\}), \\
\mathcal{B}_4 &= P_3(\{5, \dots, 10\}).
\end{aligned}$$

$\mathcal{B}_1$ and $\mathcal{B}_4$ are (2,2)-partitionable sets by the assumption. By Theorem 1, there exist LS[2]$(1, 2, 5)$ and LS[2]$(0, 1, 4)$. Therefore $\mathcal{B}_2$ and $\mathcal{B}_3$ are (2,2)-partitionable sets by Lemma 2. Now Lemma 1 shows that $P_3(X)$ is (2,2)-partitionable set, i. e. LS[2]$(2, 3, 10)$ is constructed.

The general form of the specific partition of $P_k(X)$ which appeared in the example above is as follows.

Lemma 3. Let $X = \{1, \dots, u + v + 1\}$ and let $X_j = \{1, \dots, j\}$ and $Y_j = X \setminus X_j$ for $j = 1, \dots, u + v + 1$. Assume that

$$\mathcal{B}_i = P_{k-i}(X_{u-i}) * P_i(Y_{u-i+1}), \qquad 0 \leq i \leq k.$$

Then the sets $\mathcal{B}_i$ partition $P_k(X)$.

We review the important recursive constructions obtained by the approach of (N, t)-partitionable sets in the following theorems. Let p be a prime number.

Theorem 5 [4]. If $\mathrm{LS}[N](t,i,v+i)$ exist for all $t+1 \leq i \leq k$ and $\mathrm{LS}[N](t,k,u)$ also exists, then $\mathrm{LS}[N](t,k,u+l(v+1))$ exist for all $l \geq 1$.

Theorem 6 [4]. If $\mathrm{LS}[N](t,t+1,v+t)$ exists, then $\mathrm{LS}[N](t,t+1,lv+t)$ exist for all $l \geq 1$.

Theorem 7 [2, 19]. If $\mathrm{LS}[p](t,k,v-1)$ exists, then $\mathrm{LS}[p](t,pk+i,pv+j)$ exist for all $-p \leq j < i \leq p-1$.

Theorem 8 [2]. If $\mathrm{LS}[p](t,k,v-1)$ exists, then $\mathrm{LS}[p](t+1,pk+i,pv+j)$ exist for all $0 \leq j < i \leq p-1$.

These theorems clearly have nice applications. Many infinite families of large sets can be constructed by means of these theorems. By Theorem 8, one can easily show that a large set of t-designs and therefore a t-design exists for every t, a result which was initially proved by Teirlinck [22] by a different method. As far as we know, Theorems 7 and 8 are the only known extension theorems which impose no additional conditions on the parameters.

4. Necessary Conditions

In this section, we present an alternative description of $\mathrm{B}[N](t,k)$ when N is a prime power, which we find useful in the subsequent section. We also note that it can be used for arbitrary N as well, because of the factorization of N into prime powers. Let m and n be positive integers. We denote the quotient and remainder of division m by n by $[m/n]$ and (m/n), respectively. Let p be a prime number. It is well known that the largest integer α such that $p^\alpha | m!$ is equal to $\sum_{i\geq 1}[m/p^i]$. We denote the largest value α such that

$p^\alpha | \binom{m}{n}$ by $(m,n)_p$. Therefore we have

$$(m,n)_p = \sum_{i\geq 1} \left[\frac{m}{p^i}\right] - \left[\frac{n}{p^i}\right] - \left[\frac{m-n}{p^i}\right].$$

Note that one can evaluate the expression $[m/p^i] - [n/p^i] - [(m-n)/p^i]$ in the following way.

$$\left[\frac{m}{p^i}\right] - \left[\frac{n}{p^i}\right] - \left[\frac{m-n}{p^i}\right] = \begin{cases} 1 & \text{if } \left(\frac{m}{p^i}\right) < \left(\frac{n}{p^i}\right), \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We now state the main theorem.

Theorem 9. $v \in \mathrm{B}[p^\alpha](t,k)$ if and only if there exist distinct positive integers ℓ_i for $1 \leq i \leq \alpha$ such that $t \leq (v/p^{\ell_i}) < (k/p^{\ell_i})$ for all i.

Proof. First assume that $v \in \mathrm{B}[p^\alpha](t,k)$. For $0 \leq j \leq t$ we have

$$\begin{aligned}(v-j,k-j)_p &= \sum_{r\geq 1} \left[\frac{v-j}{p^r}\right] - \left[\frac{k-j}{p^r}\right] - \left[\frac{v-k}{p^r}\right] \\ &\geq \alpha. \end{aligned} \tag{5}$$

Let ℓ_0 be the largest integer such that $(v/p^{\ell_0}) \geq t$, but $(v/p^{\ell_0-1}) < t$. Let $j_0 = (v/p^{\ell_0-1}) + 1$, if $(v/p^{\ell_0-1}) < (k/p^{\ell_0-1})$ and $j_0 = (k/p^{\ell_0-1})$, otherwise. Therefore,

$$j_0 \leq \left(\frac{k}{p^r}\right), \qquad r \geq \ell_0 - 1. \tag{6}$$

By (4), we have

$$\sum_{r=1}^{\ell_0-1} \left[\frac{v-j_0}{p^r}\right] - \left[\frac{k-j_0}{p^r}\right] - \left[\frac{v-k}{p^r}\right] = 0. \tag{7}$$

Now by (4)-(7), there exist distinct positive integers $\ell_i \geq \ell_0$ for $1 \leq i \leq \alpha$ such that $((v-j_0)/p^{\ell_i}) < ((k-j_0)/p^{\ell_i})$ or $t \leq (v/p^{\ell_i}) < (k/p^{\ell_i})$ for all i.

Now suppose that there exist distinct positive integers ℓ_i for $1 \le i \le \alpha$ such that $t \le (v/p^{\ell_i}) < (k/p^{\ell_i})$. Therefore, $((v-j)/p^{\ell_i}) < ((k-j)/p^{\ell_i})$ for all $0 \le j \le t$ and consequently $(v-j, k-j)_p \ge \alpha$ which in turn implies that $v \in \mathrm{B}[p^\alpha](t,k)$. $\square$

By Theorem 9, we are able to identify $\mathrm{B}[N](t,t+1)$ completely.

Lemma 4 [19]. Let $\prod_{i=1}^s p_i^{\alpha_i}$ be the prime power factorization of N. For $1 \le i \le s$, suppose that $p_i^{s_i-1} \le t+1 < p_i^{s_i}$. Then

$$\mathrm{B}[N](t,t+1) = \left\{ v | v \equiv t \pmod{\prod_{i=1}^{s} p_i^{\alpha_i+s_i-1}} \right\}.$$

Proof. By Theorem 9, $v \in \mathrm{B}[p_i^{\alpha_i}](t,t+1)$ if and if only $v = n_i p_i^{\alpha_i+s_i-1} + t$ for some n_i. Therefore $v \in \mathrm{B}[N](t,t+1)$ if and if only $v = n\prod_{i=1}^s p_i^{\alpha_i+s_i-1} + t$ for some n. $\square$

The following result is due to Teirlinck and we prove it by using Lemma 4.

Lemma 5 [20]. For $k = t+1$, We have

$$\lambda_{\min} = \gcd(v-t, \mathrm{lcm}(1,\ldots,t+1)).$$

Proof. Let $\prod_{i=1}^s p_i^{\alpha_i}$ be the prime power factorization of $v-t$ and let $p_i^{s_i-1} \le t+1 < p_i^{s_i}$ for $1 \le i \le s$. If $v \in \mathrm{B}[N](t,t+1)$, then N is at most equal to $\prod_{i=1}^s p_i^{\alpha_i - s_i+1}$. Therefore $\lambda_{\min} = \lambda_{\max}/N = \prod_{i=1}^s p_i^{s_i-1}$. This proves the assertion. $\square$

We bring this section to an end by presenting another useful application of Theorem 9.

Lemma 6. The minimal element of $\mathrm{B}[p^\alpha](t,k)$ is equal to $v_{\min} = ([k/p^{\ell+\alpha-1}]+1)p^{\ell+\alpha-1}+t$ in which ℓ is the smallest positive integer such that $(k/p^\ell) > t$.

Proof. Let $\ell_1 = \ell, \ell_2 = \ell+1, \ldots,$ and $\ell_\alpha = \ell+\alpha-1$. It is easy to check that $v = ([k/p^{\ell_\alpha}]+1)p^{\ell_\alpha}+t \in \mathrm{B}[p^\alpha](t,k)$. Now suppose that $v' \in \mathrm{B}[p^\alpha](t,k)$. By Theorem 9, there are distinct positive integers ℓ'_i, $1 \le i \le \alpha$, such that $t \le (v'/p^{\ell'_i}) < (k/p^{\ell'_i})$. Clearly $\ell'_i \ge \ell_i$ for all i and so we have

$$\begin{aligned} v' &= \left[\frac{v'}{p^{\ell'_\alpha}}\right] p^{\ell'_\alpha} + \left(\frac{v'}{p^{\ell'_\alpha}}\right) \\ &\ge \left(\left[\frac{k}{p^{\ell'_\alpha}}\right]+1\right) p^{\ell'_\alpha} + t \\ &\ge \left(\left[\frac{k}{p^{\ell_\alpha}}\right]+1\right) p^{\ell_\alpha} + t \\ &= v. \end{aligned}$$

Therefore, $v = v_{\min}$ and the proof is complete. $\Box$

5. Root Cases

In this section we extend recursive constructions of large sets of t-designs of prime sizes by the notion of (N,t)-partitionable sets and the approach described in Section 3. Theorem 10 shows that for given t and k there are a finite number of certain large sets which suffice to produce large sets for every possible order v. We call these large sets *root cases*. The root cases of large sets of size 2 have already been determined by Ajoodani-Namini [1]. He has also constructed them for $t = 2$ and arbitrary k. Let p be a prime and suppose that t and k are given.

Lemma 7. let ℓ be the smallest positive number such that $(k/p^\ell) > t$. Suppose that $([k'/p^\ell]+1)p^\ell + t \in \mathrm{A}[p](t,k')$ for all k' provided that $t+1 \le k' \le k$ and $(k'/p^\ell) > t$. Then $np^\ell + t \in \mathrm{A}[p](t,k)$ for all $n > [k/p^\ell]$.

Proof. The proof is by induction on k. We also proceed by induction on n. For $n = [k/p^\ell] + 1$, there is nothing to prove. So let $n > [k/p^\ell] + 1$. Let $u = (n-1)p^\ell + t$, $v = p^\ell - 1$ and let

$$\begin{aligned} X = &\ \{1, \ldots, u+v+1\}, \\ X_j = &\ \{1, \ldots, j\}, \\ Y_j = &\ X \setminus X_j, \qquad\qquad\qquad 1 \le j \le u+v+1, \\ \mathcal{B}_i = &\ P_{k-i}(X_{u-i}) * P_i(Y_{u-i+1}), \quad 0 \le i \le k. \end{aligned}$$

By Lemma 3, the sets $\mathcal{B}_i$ partition $P_k(X)$. It is sufficient to show that every $\mathcal{B}_i$ is (p,t)-partitionable. Then by Lemma 1, $P_k(X)$ will be (p,t)-partitionable and so $np^\ell + t \in \mathrm{A}[p](t,k)$. By the induction hypothesis, $\mathcal{B}_0$ is (p,t)-partitionable.

First let $1 \le i \le t$. By Theorem 1, $u - i \in \mathrm{A}[p](t-i, k-i)$. Therefore, $P_{k-i}(X_{u-i})$ is a $(p, t-i)$-partitionable set. By the assumption, we have $v+t+1 = p^\ell + t \in \mathrm{A}[p](t, t+1)$, which in turn implies that $v+i \in \mathrm{A}[p](i-1, i)$ by Theorem 1. Since $|Y_{u-i+1}| = v+i$, it is clear that $P_i(Y_{u-i+1})$ is $(p, i-1)$-partitionable. Now by Lemma 2, $\mathcal{B}_i$ is (p,t)-partitionable.

Now let $t+1 \le i \le k$. We first consider the case $(i/p^\ell) > t$. By the assumption, for $t+1 \le j \le (i/p^\ell)$ we have $([i/p^\ell]+1)p^\ell + t \in \mathrm{A}[p](t, [i/p^\ell]p^\ell + j)$. By Theorem 3, it is implied that $v + i = ([i/p^\ell]+1)p^\ell + (i/p^\ell) - 1 \in \mathrm{A}[p](t,i)$. Therefore, $P_i(Y_{u-i+1})$ is (p,t)-partitionable and by Lemma 2, $\mathcal{B}_i$ is also (p,t)-partitionable. Now consider the case $0 \le (i/p^\ell) \le t$. Then, we have $[k/p^\ell] \ge 1$. For a moment suppose that $(i/p^\ell) \ne 0$. Notice that $v+i+t+1-(i/p^\ell) = ([i/p^\ell]+1)p^\ell + t$ and $i+t+1-(i/p^\ell) = [i/p^\ell]p^\ell + t + 1$. By the assumption, we have $v+i+t+1-(i/p^\ell) \in \mathrm{A}[p](t, i+t+1-(i/p^\ell))$ which by Theorem 1, results in $v+i \in \mathrm{A}[p]((i/p^\ell)-1, i)$. Hence $P_i(Y_{u-i+1})$ is $(p, (i/p^\ell)-1)$-partitionable . We now allow that $(i/p^\ell) = 0$. Since $u - i + (i/p^\ell) = (n-1-[i/p^\ell])p^\ell + t$ and $k - i + (i/p^\ell) = ([k/p^\ell] - [i/p^\ell])p^\ell + (k/p^\ell)$, therefore, by the induction on k we obtain that $u - i + (i/p^\ell) \in \mathrm{A}[p](t, k-i+(i/p^\ell))$. Eventually, Theorem 1 yields that $u - i \in \mathrm{A}[p](t-(i/p^\ell), k-i)$ and $P_{k-i}(X_{u-i})$ is $(p, t-(i/p^\ell))$-partitionable and thus $\mathcal{B}_i$ is (p,t)-parititionable.

This completes the proof. $\square$

By Lemma 7, we can determine the root cases for given t and k.

Theorem 10. Let ℓ be the smallest positive integer such that $(k/p^\ell) > t$. Suppose that $p^\ell + t \in \mathrm{A}[p](t, k')$ for all k' provided that $t + 1 \leq k' \leq \min(k, (p^\ell + t)/2)$. Then $\mathrm{A}[p](t, k) = \mathrm{B}[p](t, k)$.

Proof. Throughout the proof, we assume that $t+1 \leq k' \leq k$. If $(p^\ell+t)/2 < k' < p^\ell$, then $p^\ell + t - k' < (p^\ell + t)/2$ and so $p^\ell + t \in \mathrm{A}[p](t, p^\ell + t - k')$. Hence by taking complement we obtain that $p^\ell + t \in \mathrm{A}[p](t, k')$. Using Lemma 7 for every $k' < p^\ell$ and $n \geq 1$ we have $np^\ell + t \in \mathrm{A}[p](t, k')$. Now let $k' > p^\ell$ and $(k'/p^\ell) > t$. We have $t < p^\ell + t - (k'/p^\ell) < p^\ell < k$ and therefore, $([k'/p^\ell] + 1)p^\ell + t \in \mathrm{A}[p](t, p^\ell + t - (k'/p^\ell))$ which by taking complement one can deduce that $([k'/p^\ell] + 1)p^\ell + t \in \mathrm{A}[p](t, k')$. By Lemma 7, for all k' and $n > [k'/p^\ell]$ we obtain that

$$np^\ell + t \in \mathrm{A}[p](t, k'). \tag{8}$$

Let $v \in \mathrm{B}[p](t, k)$. By Theorem 9, there exists $r \geq \ell$ such that $t \leq (v/p^r) < (k/p^r)$. By (8), we have

$$[v/p^r]p^r + t \in \mathrm{A}[p](t, [k/p^r]\, p^r + j),$$

for $(k/p^r) - (v/p^r) + t \leq j \leq (k/p^r)$. Hence, by Theorem 3, $v = [v/p^r]p^r + (v/p^r) \in \mathrm{A}[p](t, k)$. $\square$

An explicit form of Theorem 10 is presented in the following theorems. Their proofs are similar and hence we only present the proof of Theorem 12. Again suppose that t and k are given.

Theorem 11 [1]. Let $2^s - 1 \leq t < 2^{s+1} - 1$. Suppose that for every j and n such that $0 \leq j \leq [t/2]$ and $t + 1 \leq 2^n + j \leq k$, there exists

LS[2]$(t, 2^n + j, 2^{n+1} + t)$. Then A[2]$(t_1, k_1) =$ B[2](t_1, k_1) for all $2^s - 1 \le t_1 \le t$ and $k_1 \le k$.

Theorem 12. Let p be an odd prime and let $p^s - 1 \le t < p^{s+1} - 1$. Suppose that the following conditions hold:

(i) There exists LS[p]$(t, k', p^{s+1} + t)$ for every $t + 1 \le k' \le \min(k, (p^{s+1} + t)/2)$.

(ii) There exists LS[p]$(t, ip^n + j, p^{n+1} + t)$ for every i, j and n such that $0 \le j \le t, 1 \le i \le (p-1)/2, ip^n + j \le k$ and $n > s$.

Then A[p]$(t_1, k_1) =$ B[p](t_1, k_1) for all $p^s - 1 \le t_1 \le t$ and $k_1 \le k$.

Proof. We use an induction on $t_1 + k_1$. First let $t_1 = p^s - 1$ and $k_1 = p^s$. From LS[p]$(t, t+1, p^{s+1} + t)$ and Theorem 1 we obtain LS[p]$(t_1, k_1, p^{s+1} + t_1)$. Therefore we are done by Theorem 10. Now suppose that $2p^s - 1 < t_1 + k_1 \le t + k$ and $t_1 \le t$. By Theorem 10, and the induction hypothesis, it is sufficient to establish the existence of an LS[p]$(t_1, k_1, p^{\ell_1} + t_1)$ in which ℓ_1 is the smallest positive integer such that $(k_1/p^{\ell_1}) > t_1$ and

$$k_1 < (p^{\ell_1} + t_1)/2. \tag{9}$$

By $[k_1/p^{\ell_1}] = 0$, we have $\ell_1 \ge s + 1$. If $\ell_1 = s + 1$, then by (ii), we can obtain LS[p]$(t_1, k_1, p^{s+1} + t_1)$ from LS[p]$(t, \max(t+1, k_1), p^{s+1} + t)$ using Theorem 1. If $\ell_1 > s + 1$, then $k_1 \ge p^{s+1}$. Let $[k_1/p^{\ell_1 - 1}] = i$ and $(k_1/p^{\ell_1 - 1}) = j$. Clearly $j \le t_1 \le t$. By (9), we also obtain that $i \le (p-1)/2$. Now LS[p]$(t, ip^{\ell_1 - 1} + j, p^{\ell_1} + t)$, which exist by (ii), may be employed to obtain LS[2]$(t_1, ip^{\ell_1 - 1} + j, p^{\ell_1} + t_1)$ via Theorem 1. □

6. Existence Results

In this section we use Theorems 11 and 12 to prove some existence results on large sets of sizes 2, 3, 5, 7, 11 and 29. Many large sets with small parameters have recently been constructed by Laue, Magliveras, and Wassermann [15] among them there are some root cases. The result of Theorem 16 below is new and the other results have already appeared in the literature. Theorems 17–21 appear in [15] without proofs. Perhaps the most celebrated result in this context is the following theorem due to Ajoodani-Namini.

Theorem 13 [1]. $\mathrm{A}[2](2,k) = \mathrm{B}[2](2,k)$.

Proof. By Theorem 11, we need the classes of large sets $\mathrm{LS}[2](2,2^n,2^{n+1}+2)$ and $\mathrm{LS}[2](2,2^n+1,2^{n+1}+2)$. The second class is constructed by using Baranyai's Theorem [6, 9] and Alltop's Extension Theorem (Theorem 4). Ajoodani-Namini has also constructed the first class by use of $(2,t)$-partitionable sets or *trades* in [1]. $\square$

Theorem 14 [1, 4, 9, 14]. If $3 \le t \le 5$ and $k \le 15$ or $t = 6$ and $k = 7, 8, 9$, then $\mathrm{A}[2](t,k) = \mathrm{B}[2](t,k)$.

Proof. First suppose that $3 \le t \le 6$ and $k \le 9$. By Theorem 11, we need the following large sets:

$$\text{(i) } \mathrm{LS}[2](6,7,14), \quad \text{(ii) } \mathrm{LS}[2](6,8,22), \quad \text{(iii) } \mathrm{LS}[2](6,9,22).$$

These large sets exist by [13], [14] and [14], respectively. To complete the proof we also need $\mathrm{LS}[2](5,10,21)$ which is known to exist by [14]. $\square$

Theorem 15 [19]. If $t \le 4$ and $k \le 8$, then $\mathrm{A}[3](t,k) = \mathrm{B}[3](t,k)$.

Proof. Large sets LS[3]$(4,5,13)$ and LS[3]$(4,6,13)$ exist by [11] and [15], respectively. Therefore, by Theorem 12, we are done. $\square$

Theorem 16. If $k \leq 80$, then A[3]$(2,k)$ = B[3]$(2,k)$.

Proof. By Theorems 12, we have to establish the existence of the following large sets.

$$\begin{array}{lll} \text{(i) LS[3]}(2,9,29), & \text{(ii) LS[3]}(2,10,29), & \text{(iii) LS[3]}(2,11,29), \\ \text{(iv) LS[3]}(2,27,83), & \text{(v) LS[3]}(2,28,83), & \text{(vi) LS[3]}(2,29,83). \end{array}$$

There exist LS[3]$(2,9,29)$ and LS[3]$(3,11,30)$ by [15]. By the last large set we obtain LS[3]$(2,10,29)$, LS[3]$(2,11,29)$. Large sets (iv)-(vi) are constructed by Theorem 6 in [15]. $\square$

Theorem 17 [15]. If $k \leq 5$, then A[5]$(2,k)$ = B[5]$(2,k) \setminus \{7\}$.

Proof. By Theorem 9, we have

$$\begin{aligned} \text{B[5]}(2,3) &= \{5l+2 |\ l \geq 1\}, \\ \text{B[5]}(2,4) &= \{5l+i |\ l \geq 1, \quad i = 2,3\}. \end{aligned}$$

It is well known that LS[5]$(2,3,7)$ and LS[5]$(2,4,7)$ do not exist. Since LS[5]$(2,3,12)$ and LS[5]$(2,3,17)$ exist [24], we obtain LS[5]$(2,3,5l{+}2)$ $(l \geq 2)$ by Theorem 5.

There exist large sets LS[5]$(2,4,8)$ [18], LS[5]$(2,4,12)$ [12], LS[5]$(2,4,13)$ [8] and LS[5]$(2,4,17)$ [23]. Therefore, by Theorem 5, we are able to construct LS[5]$(2,4,5l+i)$ for all $l \geq 2$ and $i = 2,3$.

For $k = 5$ we use Theorem 10. It suffices to have LS[5]$(2,3,27)$, LS[5]$(2,4,27)$ which exist by the paragraphs above and LS[5]$(2,5,27)$ which exists by [15]. $\square$

Theorem 18 [15]. If $k \leq 5$, then $\mathrm{A}[5](3,k) = \mathrm{B}[5](3,k) \setminus \{8\}$.

Proof. By Theorem 9, we have

$$\mathrm{B}[5](3,4) = \{5l+3 |\ l \geq 1\}.$$

LS[5](3, 4, 8) does not exist because of non-existence of LS[5](2, 3, 7). Since LS[5](3, 4, 13) [11] and LS[5](3, 4, 18) [23] exist, we obtain LS$[5](3,4,5l+3)$ $(l \geq 2)$ by Theorem 5.

For $k = 5$ we use Theorem 10. It suffices to have LS[5](3, 4, 28) which exist by the paragraph above and LS[5](3, 5, 28) which exists by [15]. □

Theorem 19 [15]. If $k \leq 6$, then $\mathrm{A}[7](2,k) = \mathrm{B}[7](2,k)$.

Proof. By Theorem 12, we need LS[7](2, 3, 9) and LS[7](2, 4, 9) which exist by [10] and [12], respectively. □

Theorem 20 [15]. If $k \leq 10$, then $\mathrm{A}[11](2,k) = \mathrm{B}[11](2,k)$.

Proof. By Theorem 12, we need LS[11](2, 3, 13), LS[11](2, 4, 13), LS[11](2, 5, 13) and LS[11](2, 6, 13) which exist by [17], [8], [7] and [7], respectively. □

Theorem 21 [15]. If $k \leq 5$, then $\mathrm{A}[29](2,k) = \mathrm{B}[29](2,k)$.

Proof. By Theorem 12, we need LS[29](2, 3, 31), LS[29](2, 4, 31) and LS[29](2, 5, 31) which exist by [17], [15] and [15], respectively. □

An almost comprehensive table of small large sets for $v \leq 18$ is given in [7]. We present a resume of known results on infinite cases in Table I.

Table I

N	t	k	v	Ref.
$*$	1	$*$	$*$	[6, 9]
$*$	2	3	$\neq 7$	[16, 17, 21]
2	2	$*$	$*$	[1]
2	≤ 5	≤ 16	$*$	[1, 4, 9, 14]
2	6	$7, 8, 9$	$*$	[1, 14]
3	2	≤ 80	$*$	This paper
3	≤ 4	≤ 8	$*$	[19]
5	2	≤ 5	$\neq 7$	[15]
5	3	≤ 5	$\neq 8$	[15]
7	2	≤ 6	$*$	[15]
11	2	≤ 10	$*$	[15]
29	2	≤ 5	$*$	[15]

$*$ All feasible values.

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References

1. S. Ajoodani-Namini, *All block designs with* $b = \binom{v}{k}/2$ *exist*, Discrete Math. **179** (1998), 27–35.

2. S. Ajoodani-Namini, *Extending large sets of t-designs*, J. Combin. Theory Ser. A **76** (1996), 139–144.

3. S. Ajoodani-Namini and G. B. Khosrovshahi, *On a conjecture of A. Hartman*, Combinatorics advances (Tehran), 1994, pp. 1–12, Math. Appl. 329, Kluwer Acad. Publ., Dordrecht, 1995.

4. S. Ajoodani-Namini and G. B. Khosrovshahi, *More on halving the complete designs*, Discrete Math. **135** (1994), 29–37.

5. W. O. Alltop, *Extending t-designs*, J. Combin. Theory Ser. A **18** (1975), 177–186.

6. Z. Baranyai, *On the factorizations of the complete uniform hypergraph*, Finite and infinite sets, Colloq. Math. Soc., Janos Bolyai, Vol. 10, North-Holland, Amsterdam (1975), pp. 91–108.

7. Y. M. Chee and S. S. Magliveras, *A few more large sets of t-designs*, J. Combin. Des. **6** (1998), 293–308.

8. L. G. Chouinard II, *Partitions of the* 4*-subsets of a* 13*-set into disjoint projective planes*, Discrete Math. **45** (1983), 297–300.

9. A. Hartman, *Halving the complete design*, Ann. Discrete Math. **34** (1987), 207–224.

10. T. P. Kirkman, *Note on an unanswered prize question*, Cambridge and Dublin Math. J. **5** (1850), 255–262.

11. E. S. Kramer, S. S. Magliveras, and E. A. O'Brien, *Some new large sets of t-designs*, Australas. J. Combin. **7** (1993), 189–193.

12. E. S. Kramer, S. S. Magliveras, and D. R. Stinson, *Some small large sets of t-designs*, Australas. J. Combin. **3** (1991), 191–205.

13. D. L. Kreher and S. P. Radziszowski, *The existence of simple* 6-$(14,7,4)$ *designs*, J. Combin. Theory Ser. A **43** (1986), 237–243.

14. R. Laue, *Halvings on small point sets*, J. Combin. Des. **7** (1999), 233–241.

15. R. Laue, S. S. Magliveras, and A. Wassermann, *New large sets of t-designs*, preprint.

16. J. X. Lu, *On large sets of disjoint Steiner triple systems I, II, III*, J. Combin. Theory Ser. A **34** (1983), 140–146, 147–155, and 156–182.

17. J. X. Lu, *On large sets of disjoint Steiner triple systems IV, V, VI*, J. Combin. Theory Ser. A **37** (1984), 136–163, 164–188, and 189–192.

18. M. J. Sharry and A. P. Street, *Partitioning sets of quadruples into designs I*, Discrete Math. **77** (1989), 299–305.

19. B. Tayfeh-Rezaie, *Some infinite families of large sets of t-designs*, J. Combin. Theory Ser. A **87** (1999), 239–245.

20. L. Teirlinck, *Large sets of disjoint designs and related structures*, Contemporary design theory, A collection of surveys (J. H. Dinitz and D. R. Stinson, Eds.), pp. 561–592, Wiley, New York, 1992.

21. L. Teirlinck, *A completion of Lu's determination of the spectrum for large sets of Steiner triple systems*, J.Combin. Theory Ser. A **57** (1991), 302–305.

22. L. Teirlinck, *Nontrivial t-designs without repeated blocks exist for all t*, Discrete Math. **65** (1987), 301–311.

23. L. Teirlinck, *On large sets of disjoint quadruple systems*, Ars Combin. **17** (1984), 173–176.

24. L. Teirlinck, *On the maximum number of disjoint triple systems*, J. Geom. **6** (1975), 93–96.