On skew-regular Hadamard matrices

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Abstract

Skew-regular Hadamard matrices are introduced as skew-type Hadamard matrices for which the absolute value of the row sums are constant. It is shown that there are at least 157132 skew-regular Hadamard matrices of order 36 and none of order $16m^2$, m a positive integer. The implications are significant.

1 Introduction

A matrix W of order n with $p \leq n$ nonzero entries in $\{-1, 1\}$ in each row and column and mutually orthogonal rows (and columns) is called a *weighing* matrix denoted by $W(n, p)$. A weighing matrix $W(n, p)$ with $p = n$ is a *Hadamard* and with $p = n - 1$ a *conference* matrix. A Hadamard matrix is *regular* if it has a constant row sum. The order of a regular Hadamard matrix is a perfect square n^2, and the row and column sums are n. The Hadamard matrix H is said to be *skew-type* if $H + H^T = 2I$. A *skew-regular Hadamard* matrix is a skew-type Hadamard matrix for which the

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absolute value of row sums is constant. Skew-type and regular Hadamard matrices are two distinguished classes with multiple applications; see [3, 5]. It is conjectured that there is a skew-type Hadamard matrix of order $4n$ and a regular Hadamard matrix of order $4n^2$ for any positive integer n. Skew-regular Hadamard matrices inherit properties of both skew-type and regular Hadamard matrices and, as such, lead to some powerful results. The primary reference for the article is [3].

2 Non-Existence

The following is well-known.

Lemma 1. *Let H be a Hadamard matrix of order n with column-sums c_i. Then $\sum c_i^2 = n^2$.*

Proof. Let j be the all-ones column vector.

$$\sum c_i^2 = (c_1, \ldots, c_n)(c_1, \ldots, c_n)^T = (j^T H)(j^T H)^T = j^T(HH^T)j = j^T(nI)j = n^2.$$

□

In 1977, Best [2] showed that the column sums of a Hadamard matrix of order n are the same if and only if they are all equal to $\pm\sqrt{n}$.

Lemma 2. *Let H be a skew-regular Hadamard matrix of order $4m^2$. Then there are $2m^2 - m$ rows with row-sum $-2m$.*

Proof. Let k be the number of rows with row-sum $-2m$. Since H is skew-regular, the remaining $4m^2 - k$ rows have row-sum $2m$. Moreover, since H is skew-type, there are exactly k columns with column-sum $2m + 2$ and $4m^2 - k$ columns with column-sum $-2m + 2$. By Lemma 1, we have that $k(2m+2)^2 + (4m^2 - k)(-2m+2)^2 = 16m^4$. Solving for k, we have the desired result. □

Theorem 3. *Let H be a skew-regular Hadamard matrix of order $4m^2$. Then m is odd.*

Proof. Note that each column sum of H is $2m+2$ modulo 4 and that negating any row of H changes every column sum by exactly two modulo 4. By Lemma 2, there are exactly $2m^2 - m$ rows with row-sum $-2m$. If we negate each of these rows to obtain H', the column-sums of H are changed by $2(2m^2 - m)$ modulo 4. Since each column-sum of H was originally $2m+2$ modulo 4, the new column-sums are all 2 modulo 4. Since H' is regular, the column-sums must equal $\pm 2m$. Thus, m is odd. $\square$

We now have a simple observation which has some significant implications.

Theorem 4. *Let H be a Hadamard matrix of order $4m^2$. H is equivalent to a skew-regular Hadamard matrix if and only if it is equivalent to a regular Hadamard matrix and a skew-type Hadamard matrix.*

Proof. The first implication is quite simple. Assume that H is equivalent to a skew-regular Hadamard matrix K. We may negate appropriate rows of K to arrive at a regular Hadamard matrix, and K itself is skew-type.

The second implication is trickier. Assume that H is equivalent to a skew-type Hadamard matrix S and a regular Hadamard matrix R. Note that permuting the rows and columns of a Hadamard matrix does not change row or column sums. Thus, permute the rows and columns of R so that $R = PSQ^T$ for some signed permutation matrices P, Q. Note that the row-sums of $SQ^T (= P^T R)$ are all $\pm 2m$, and thus, so are QSQ^T. Since S is skew-type, QSQ^T is also skew-type. Thus, QSQ^T is a skew-regular Hadamard matrix equivalent to H. $\square$

The following are some immediate corollaries.

Corollary 5. *Let H be a skew-type Hadamard matrix of order n. If $n \neq 4(2k+1)^2$ and $n \neq 1$, then H is not equivalent to a regular Hadamard matrix.*

Corollary 6. *Let H be a regular Hadamard matrix of order n. If $n \neq 4(2k+1)^2$ and $n \neq 1$, then H is not equivalent to a skew-type Hadamard matrix.*

The existence of skew-type Hadamard matrices of order $16m^2$ are known for infinitely many m [3], and so none is equivalent to a regular Hadamard matrix. The conclusion is interesting even for Hadamard matrices as small as order 16.

3 Existence

We begin with the only example of a skew-regular Hadamard matrix of order a power of 2, namely, of order 4. Throughout -1 is shown by $-$.

Example 7.

$$\begin{bmatrix} 1 & 1 & - & 1 \\ - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ - & - & - & 1 \end{bmatrix}$$

This example led to more searches for Hadamard matrices with similar properties, and not only one but many of order 36 was constructed.

Example 8. We start with one example of an order of 36.

$$
\begin{bmatrix}
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-&-&-&-&1&1&-&1&-&-&1&-&1&1&-&1&1&1&1&-&1&1&1&-&1&1&1&-&-&-&-&1&1&1&1&1
\end{bmatrix}
$$

Two skew Hadamard matrices are considered SH-equivalent if they are similar by a signed permutation matrix. In [1], a classification of SH-inequivalent skew-type Hadamard matrices of order 36 for some types was given and a total of 157132 SH-inequivalent skew-type Hadamard matrices of order 36 were found. Here, we try to find which of those are SH-equivalent to a skew-regular Hadamard matrix. Since permutations on rows or columns do not change row and column sums, we only need to consider signed permutation matrices with nonzero entries only on the main diagonal. A straightforward check, i.e., examining all such signed permutation matrices of order 36, is computationally time-consuming as there are 2^{36} such matrices. Our experiment shows that the running time for each 157132 matrix is about one to two minutes on a single desktop computer. To make it faster, we pursue a different approach. Let H be a skew-type Hadamard matrix of order 36. We are looking for a signed permutation matrix P with nonzero entries only on the main diagonal such that $P^T HP$ has row sums 6 or -6. Let j_n be all one column vector of dimension n. Then $H(Pj_{36})$ is a column vector with entries 6 or -6 which means that $H(Pj_{36}) = 0 \pmod 3$. This shows that Pj_{36}, which is a $(-1, 1)$ column vector, is contained in the null space of H over the three element finite field $GF(3)$. Note that the dimension of the null space of H over $GF(3)$ is 18; see [1]. So instead of 2^{36} candidates for Pj_{36}, we have 2^{18} candidates coming from the null space of H over $GF(3)$. We find a standard basis for the null space of H over $GF(3)$ and then examine all 2^{18} linear combinations of the vectors in the basis with coefficients -1 and 1 as candidates for Pj_{36}. This approach is much faster than the initially straightforward approach. It took only 10 minutes to examine all 157132 matrices on a single desktop computer. The results show that every matrix in the list of 157132 matrices is SH-equivalent to a skew-regular Hadamard matrix. We have the following.

Theorem 9. *There are at least* 157132 *skew-regular Hadamard matrices of order* 36*.*

The 157132 skew-regular Hadamard matrices of order 36 are available upon request.

The next Hadamard matrix of order $4m^2$, m odd, is order 100, which motivates us to raise the following questions.

- Is it true that, as it may be the case for order 36, any skew-type Hadamard matrix of order 100 is equivalent to a regular one?
- Is it true that any skew-type Hadamard matrix of order $4(2k + 1)^2$, k any positive integer, is equivalent to a regular one?
- Are there more than 157132 inequivalent skew-regular Hadamard matrices of order 36?

4 Applications

This section connects skew-regular Hadamard matrices, biregular skew conference matrices, and doubly regular tournaments with a two-intersection set. Moreover, we show that a skew-regular Hadamard matrix gives rise to a conference matrix with maximum excess; refer to [4] for details.

For a $(0, 1, -1)$-matrix W, the sum of entries of W, denoted $E(W)$, is called the *excess* of the matrix W. An upper bound of the excess of conference matrices of order n, $n-1$ a non-square, is known as follows.

Proposition 10. [4, Proposition 7] *Let W be a conference matrix of order n with $n-1$ a non-square. Let k be an odd integer such that $k \leq \sqrt{n-1} < k+2$. Then*

$$E(W) \leq \frac{n(k^2+2k+n-1)}{2(k+1)}$$

with equality holds if and only if Wj_n has entries $k, k+2$.

A two-intersection set with parameters $(k; \alpha, \beta)$ for a 1-design $(P, \mathcal{B})$ is a k-subset D of P such that the set

$$\{|B \cap D| : B \in \mathcal{B}\}$$

contains exactly two numbers α and β. That is to say, letting N be the incidence matrix of the design $(P, \mathcal{B})$, there exists a two-intersection set with parameters $(k; \alpha, \beta)$ if and only if there exist $(0, 1)$-vectors x, y indexed by the elements of P such that $j_{|P|}^T x = k$, $x^T N = \alpha y^T + \beta(j_{|P|}^T - y^T)$.

A tournament is regarded as a design with $|P| = |\mathcal{B}|$ and incidence matrix N satisfying $N + N^T = J - I$. A tournament of order $4t+3$ is doubly regular if its adjacency matrix A satisfies that $AA^T = (t+1)I + tJ$.

Proposition 11. *Let $n = 4m^2 - 1$, $m \in \mathbb{N}$, be an order for a doubly regular tournament. If a two-intersection set with parameters $(2m^2 + m; m^2, m^2 + m)$ exists for a doubly regular tournament of order n, then a skew-regular Hadamard matrix of order $4m^2$ exists.*

Proof. Let A be the adjacency matrix of a doubly regular tournament of order n. It is easy to see that the matrix $C = \begin{pmatrix} 0 & j_n^T \\ -j_n & A - A^T \end{pmatrix}$ is a conference matrix of order $n+1$.

Let x, y be $(0,1)$-vectors so that $A^T x = \alpha y + \beta(j_n - y)$. Let D be a diagonal matrix of order n with diagonal entries $D_{ii} = -2x_i + 1$, and $D' = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$. We claim that $D'CD'$ is a skew-regular conference matrix of order $n+1$. First we calculate CD' as follows:

$$\begin{aligned}
CD' j_{n+1} &= \begin{pmatrix} 0 & -2x^T + j_n^T \\ -j_n & (A - A^T)D \end{pmatrix} \begin{pmatrix} 1 \\ j_n \end{pmatrix} \\
&= \begin{pmatrix} 2x^T j_n - j_n^T j_n \\ -j_n + (A - A^T)(-2x + j_n) \end{pmatrix} \\
&= \begin{pmatrix} -2m - 1 \\ -j_n - 2(J - I - A^T)x \end{pmatrix} \\
&= \begin{pmatrix} -2m - 1 \\ -j_n - 2Jx + 2x + 2Ax \end{pmatrix} \\
&= \begin{pmatrix} -2m - 1 \\ (-2|D| - 1)j_n + 2x + 4(\alpha y + \beta(j_n - y)) \end{pmatrix} \\
&= \begin{pmatrix} -2m - 1 \\ (2m - 1)j_n + 2x - 4my \end{pmatrix} \\
&= \begin{pmatrix} -2m - 1 \\ 2m(j_n - 2y) + 2x - j_n \end{pmatrix}.
\end{aligned}$$

Since $D(2x - j_n) = -j_n$, we have

$$D'CD' = \begin{pmatrix} -2m - 1 \\ 2mD(j_n - 2y) - j_n \end{pmatrix}.$$

Since $D(j_n - 2y)$ is a $(1, -1)$-vector, $D'CD'$ has row sums $\pm 2m - 1$. Therefore $D'CD'$ is a skew-regular conference matrix of order $4m^2$

Then $H := D'CD' + I$ is a skew-regular Hadamard matrix of order $4m^2$. □

Question 12. Which doubly regular tournaments of order $4m^2 - 1$ have a two-intersection set with parameters $(2m^2 + m; m^2, m^2 + m)$?

Theorem 13. (i) *The existence of the following is equivalent.*

(a) A skew-regular Hadamard matrix of order $4m^2$.

(b) A skew conference matrix of order $4m^2$ with row sums $2m-1, -2m-1$.

(c) *A doubly regular tournament of order* $4m^2-1$ *with a two-intersection set with parameters* $(2m^2+m; m^2, m^2+m)$.

(ii) *If a skew-regular Hadamard matrix of order* $4m^2$ *exists, then a conference matrix of order* $4m^2$ *with maximal excess exists.*

Proof. (i) Set $n = 4m^2 - 1$.

(a) $\Leftrightarrow$ (b): Assume (a) holds. Let H be a skew-regular Hadamard matrix of order $4m^2$. Then, after permuting rows and columns simultaneously, we may assume that.

$$Hj_{4m^2} = \begin{pmatrix} 2mj_{2m^2+m} \\ -2mj_{2m^2-m} \end{pmatrix}.$$

Define $C = H - I$. Then C is a skew conference matrix of order $4m^2$ and

$$Cj_{4m^2} = \begin{pmatrix} (2m-1)j_{2m^2+m} \\ (-2m-1)j_{2m^2-m} \end{pmatrix},$$

which implies that (2) holds. The converse follows from reversing the argument above.

(b) $\Leftrightarrow$ (c): Assume that (c) holds. Let A be the adjacency matrix of a doubly regular tournament of order n. It is easy to see that the matrix $C = \begin{pmatrix} 0 & j_n^T \\ -j_n & A - A^T \end{pmatrix}$ is a conference matrix of order $n+1$.

Let x, y be $(0,1)$-vectors so that $A^T x = m^2 y + (m^2+m)(j_n - y), x^T j_n = 2m^2 + m$. Let D be a diagonal matrix of order n with diagonal entries $D_{ii} = -2x_i + 1$, and $D' = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$. We claim that $D'CD'$ is a skew-regular conference matrix of order $n+1$. First we calculate $CD'j_{n+1}$. We use $j_n^T D j_n = -2j_n^T x + j_n^T j_n = -2m - 1$ and $(A - A^T)Dj_n = 2m(j_n - 2y) + 2x$ to obtain

$$\begin{aligned} CD'j_{n+1} &= \begin{pmatrix} 0 & j_n^T \\ -j_n & A - A^T \end{pmatrix} \begin{pmatrix} 1 \\ Dj_n \end{pmatrix} \\ &= \begin{pmatrix} j_n^T D j_n \\ -j_n + (A - A^T)Dj_n \end{pmatrix} \\ &= \begin{pmatrix} -2m - 1 \\ 2m(j_n - 2y) + 2x - j_n \end{pmatrix}. \end{aligned}$$

Since $D(2x - j_n) = -j_n$, we have

$$D'CD' = \begin{pmatrix} -2m-1 \\ 2mD(j_n - 2y) - j_n \end{pmatrix}.$$

Since $D(j_n - 2y)$ is a $(1,-1)$-vector, $D'CD'$ has row sums $\pm 2m-1$. Therefore, $D'CD'$ is an absolutely regular symmetric conference matrix of order $4m^2$. Then $H := D'CD' + I$ is a skew-regular Hadamard matrix of order $4m^2$.

The converse follows from reversing the argument above.

(ii) Let C be a skew conference matrix of order $4m^2$ such that $Cj_{4m^2} = \begin{pmatrix} (2m-1)j_{2m^2+m} \\ (-2m-1)j_{2m^2-m} \end{pmatrix}$. Let C' be the matrix obtained from C by negating the last $2m^2 - m$ rows. Then C' is a conference matrix, and the excess of C' is

$$(2m-1)(2m^2+m) + (2m+1)(2m^2-m) = 8m^3 - 2m,$$

which attains the upper bound in Proposition 10.

$\square$

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References

[1] M. Araya, M. Harada, H. Kharaghani, A. Mohammadian B. Tayfeh-Rezaie, On the classification of skew Hadamard matrices of order 36 and their codes, *Des. Codes Cryptogr.*, to appear.

[2] M. R. Best, The excess of a Hadamard matrix. Nederl. Akad. Wetensch. Proc. Ser. A 80. *Indag. Math.* **39** (1977), no. 5, 357–361.

[3] C. J. Colbourn and J. H. Dinitz, editors. *Handbook of combinatorial designs*. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, second edition, 2007.

[4] K. Momihara, S. Suda, Conference matrices with maximum excess and two-intersection sets, *Integers* **17** (2017), Paper No. A30, 15pp.

[5] J. Seberry, M. Yamada, Hadamard matrices, sequences, and block designs, Contemporary design theory, Wiley, New York, pp. 431–560.

[6] J. J. Seidel, A survey of two-graphs. In: Colloquio Internazionale sulle Teorie Combinatorie (Proceedings, Rome, 1973), Vol. I, pp. 481–511.