On the spectrum of simple $\mathrm{T}(2, 3, v)$ trades[1]

M. Behbahani[a] G. B. Khosrovshahi[a,b,2] B. Tayfeh-Rezaie[a]

[a] *Institute for Studies in Theoretical Physics and Mathematics (IPM)*
P.O. Box 19395-5746*, Tehran, Iran*
[b] *School of Mathematics, Statistics and Computer Science,*
University of Tehran, Tehran, Iran

Abstract

In this paper, we determine the spectrum (the set of all possible volumes) of simple $\mathrm{T}(2, 3, v)$ trades for any even foundation size v.

Keywords: trades, spectrum, product of trades
MR Subject Classification: 05B07, 05B05

1 Introduction

The determination of the set of all possible volumes of $\mathrm{T}(t, k, v)$ trades, the *spectrum*, has been studied by different authors. The spectrum of a trade can be considered for a fixed (varying) foundation size of trades. Trades also could be simple or with repeated blocks. To review the background of the subject, the reader is referred to [2, 9] and the references therein.

The spectrum of $\mathrm{T}(2, 3, v)$ trades with repeated blocks has been dealt with in [7]. The aim of this paper is to consider the spectrum of simple $\mathrm{T}(2, 3, v)$ trades with given foundation size. The spectrum of Steiner $\mathrm{T}(2, 3, v)$ trades of a fixed foundation size is given in [3]. For a given foundation size v, this only constitutes about $v^2/v^3 = 1/v$ of the whole case and therefore a large part of the spectrum remains unattained. In this paper we provide the answer for v even. The main tool achieving this is the product of trades. To be more specific, we first determine the spectrum of $\mathrm{T}(1, 2, v)$ trades. Then these trades are extended to $\mathrm{T}(2, 3, v)$ trades via an extension theorem which is provided by the notion of products of trades.

[1] This research was supported in part by a grant from IPM.
[2] Corresponding author, email: rezagbk@ipm.ir.

2 Definitions and Preliminaries

Let v, k and t be integers such that $v \geq k \geq t \geq 0$ and let X be a v-set. A $T(t,k,v)$ *trade* T on X is a set of two disjoint collections T^+ and T^- of k-subsets of X (called *blocks*) such that for every t-subset A of X, the number of occurrences of A in T^+ and T^- are equal. The *foundation* of T is the set of elements of X covered by T^+ (T^-) and is denoted by $found(T)$. The number of blocks in T^+ (T^-) is called the *volume* of T denoted by $vol(T)$. If there is no repeated block in T^+ (T^-), then T is said to be *simple*. Here, we will be only concerned with simple trades. We denote a simple $\mathrm{T}(2,3,v)$ trade of foundation size u and volume s by $\mathrm{T}(s,u)$. By the notation $s = a_1, a_2, \ldots, a_r$, we will mean that s can take any of the values of $a_1, a_2, \ldots, a_r$. The following useful lemma is trivial.

Lemma 1 *Let $a_1, \ldots, a_r, b, c$ be integers such that $a_i \geq a_{i-1}$ and $c-b \geq a_i - a_{i-1} - 1$ for $2 \leq i \leq r$. If $s_1 = a_1, \ldots, a_r$ and $s_2 = b, b+1, \ldots, c$, then $s_1 + s_2 = a_1+b, a_1+b+1, a_1+b+2, \ldots, a_r+c$.*

Let $v \geq 6$ be even. We define $m(v)$ and $M(v)$ as

$$m(v) = 2\lceil v/3 \rceil, \tag{1}$$

and

$$M(v) = \begin{cases} v(v-1)(v-2)/12 & \text{if } v \equiv 2 \pmod 4, \\ v(v+1)(v-4)/12 & \text{if } v \equiv 0 \pmod 4. \end{cases} \tag{2}$$

We summarize some of the known results about the spectrum of $\mathrm{T}(2,3,v)$ trades in the following lemmas.

Lemma 2 **[3, 6, 11]** *If there is a $T(s,v)$, then $m(v) \leq s \leq M(v)$. Moreover, there is a $T(m(v)+1, v)$ if and only if $v \not\equiv 0 \pmod 6$.*

Lemma 3 **[5, 10]** *Let $6 \leq v \leq 8$. Then, there is a $T(s,v)$ if and only if one of the following occurs:*

(i) $v = 6, \quad s = 4, 6, 10;$

(ii) $v = 7, \quad s = 6, 7, 9, 10, 12;$

(iii) $v = 8, \quad s = 6, 7, \ldots, 22, 24.$

We now define the product of trades. This definition is a special case of a general notion which was originally introduced in [1]. Let X_1 and X_2 be two disjoint sets of cardinalities v_1

and v_2, respectively. For $\mathcal{B}_1 \subseteq P_{k_1}(X_1)$ and $\mathcal{B}_2 \subseteq P_{k_2}(X_2)$, we let

$$\mathcal{B}_1 * \mathcal{B}_2 = \{A_1 \cup A_2 | A_1 \in \mathcal{B}_1, \quad A_2 \in \mathcal{B}_2\}.$$

Let $T_i = \{T_i^+, T_i^-\}$ be a $\mathrm{T}(t_i, k_i, v_i)$ trade of volume s_i on X_i for $i = 1, 2$. Then, a *product* of T_1 and T_2, denoted by $T_1 * T_2$, is defined as

$$T_1 * T_2 = \{(T_1^+ * T_2^+) \cup (T_1^- * T_2^-), (T_1^+ * T_2^-) \cup (T_1^- * T_2^+)\}.$$

There is an important extension theorem concerning the products of trades which is presented in the following theorem.

Theorem 1 [1] *$T_1 * T_2$ is a $\mathrm{T}(t_1 + t_2 + 1, k_1 + k_2, v_1 + v_2)$ trade of volume $2s_1s_2$.*

This theorem suggests that $\mathrm{T}(2, 3, v)$ trades may be constructed from $\mathrm{T}(1, 2, v)$ trades. Therefore, it is natural to determine the possible volumes for $\mathrm{T}(1, 2, v)$ trades. Let

$$S(v) = \begin{cases} v(v-2)/4 & \text{if } v \equiv 0 \pmod 2, \\ v(v-1)/4 & \text{if } v \equiv 1 \pmod 4, \\ (v+2)(v-3)/4 & \text{if } v \equiv 3 \pmod 4. \end{cases}$$

Lemma 4 *There is a $\mathrm{T}(1, 2, v)$ trade of volume s if and only if $2 \leq s \leq S(v)$, and $s \neq S(v) - 1$ in case $v \equiv 1 \pmod 4$.*

Proof It is straightforward to show that if there is a $\mathrm{T}(1, 2, v)$ trade of volume s, then $2 \leq s \leq S(v)$, and $s \neq S(v) - 1$ when $v \equiv 1 \pmod 4$. The proof of existence is by induction on v. The assertion can easily be checked for $v < 6$. Hence, suppose that $v \geq 6$. Let $X = \{x_1, \ldots, x_v\}$ and $X_i = \{x_1, \ldots, x_i\}$ for $1 \leq i \leq v$.

First suppose that $v \equiv 0 \pmod 2$ or $v \equiv 3 \pmod 4$. Let $T_1 = \{\{x_1\}, \{x_2\}\}$ and T_2 be a $\mathrm{T}(0, 1, v-2)$ trade of volume s_2 on $X \setminus X_2$. Then by Theorem 1, $T_1 * T_2$ is a $\mathrm{T}(1, 2, v)$ trade of volume $2s_2$. Let T_3 be a $\mathrm{T}(1, 2, v-2)$ trade of volume s_3 on $X \setminus X_2$. Then the union of $T_1 * T_2$ and T_3 gives a $\mathrm{T}(1, 2, v)$ trade of volume $s = 2s_2 + s_3$. If $v \equiv 0 \pmod 2$, then $s_2 = 0, 1, \ldots, (v-2)/2$, $s_3 = 2, 3, \ldots, S(v-2)$ and by Lemma 1, we obtain $s = 2, 3, \ldots, S(v)$. Now assume that $v \equiv 3 \pmod 4$. We have $s_2 = 0, 1, \ldots, (v-3)/2$ and $s_3 = 2, 3, \ldots, S(v-2) - 2, S(v-2)$. By Lemma 1, we obtain $s = 2, 3, \ldots, S(v) - 2, S(v)$. A trade of volume $S(v) - 1$ can easily be obtained from a trade T of volume $S(v)$ constructed as above. Without loss of generality, we can assume that $x_1x_3, x_2x_4 \in T^+$ and $x_3x_4 \in T^-$. We remove x_1x_3 and x_2x_4 from T^+ and x_3x_4 from T^- and add x_1x_2 to T^+. Then we find a trade of the desired volume.

The case $v \equiv 1 \pmod 4$ is dealt with in a similar way. We define trades T_i of volume s_i for $1 \leq i \leq 4$ as follows: Let $T_1 = \{\{x_1, x_2\}, \{x_3, x_4\}\}$ and T_2 be a $\mathrm{T}(0, 1, v-5)$ trade on

$X \setminus X_5$. Then $T_1 * T_2$ is a $\mathrm{T}(1,2,v-1)$ trade of volume $4s_2$. Let T_3 and T_4 be $\mathrm{T}(1,2,5)$ and $\mathrm{T}(1,2,v-4)$ trades on X_5 and $X \setminus X_4$, respectively. Then the union of $T_1 * T_2$, T_3, and T_4 gives a $\mathrm{T}(1,2,v)$ trade of volume $s = 4s_2 + s_3 + s_4$. We have $s_2 = 0, 1, \ldots, (v-5)/2$, $s_3 = 0, 2, 3, 5$ and $s_4 = 2, 3, \ldots, S(v-4) - 2, S(v-4)$. Hence, by Lemma 1, we obtain $s = 2, 3, \ldots, S(v) - 2, S(v)$, as required. $\square$

3 Nonexistence results

In this section, we show that some special volumes do not occur in the spectrum of $\mathrm{T}(2,3,v)$ trades.

Lemma 5 *If $v \equiv 0 \pmod 4$, then there is no $T(M(v)-1, v)$.*

Proof Let X be a v-set and suppose that there exists a $\mathrm{T}(M(v)-1, v)$ trade on X. Let $S = P_3(X) \setminus (T^+ \cup T^-)$. Then $|S| = v + 2$. Let r_x be the number of occurrences of point x in S. For each point x, r_x is odd and at least 3. Therefore, there are at most three points x with $r_x > 3$. The number of occurrences of each pair in S is even, so is either 0 or 2. Let a be a point such that $r_a = 3$. Without loss of generality, we suppose that $\{abc, abd, acd\} \subseteq S$ for some points b, c, d. Assume that $r_b = 3$. Then $bcd \in S$ and so $r_b = r_c = r_d = 3$. We collect all the blocks of this type in S'. Then $S \setminus S'$ must contain 6 blocks on 4 points which is impossible. $\square$

Lemma 6 *If $v \equiv 2 \pmod 4$, then there is no $T(s,v)$ for $s = M(v)-5, M(v)-3, M(v)-2, M(v)-1$.*

Proof Let X be a v-set and suppose that $T = \mathrm{T}(s,v)$ is a trade on X such that $s \in \{M(v)-5, M(v)-3, M(v)-2, M(v)-1\}$. Let $S = P_3(X) \setminus (T^+ \cup T^-)$. Then $|S| \in \{2, 4, 6, 10\}$. Let r_x and r_{xy} be the numbers of occurrences of point x and pair xy in S, respectively. Clearly, r_x and r_{xy} are even and also if $r_x \neq 0$, then $r_x \geq 4$. Hence, the number of points x such that $r_x \neq 0$ is at least 5. It yields that $|S| > 6$ and so $|S| = 10$. Since $\sum_{x \in X} r_x = 30$, there must be a point a such that $r_a = 6$ or 10. But obviously $r_a = 10$ is impossible. So $r_a = 6$. Now we show that for any pair xy, $r_{xy} \leq 2$. It is clear that $r_{xy} \leq 4$. Suppose that $r_{xy} = 4$ and $xyu, xyv, xyw, xyz \in S$. Note that $a \in \{x, y, u, v, w, z\}$ (conversely, r_u is odd, a contradiction). Assume that $a = u$ and let $B \in S \setminus \{xyu, xyv, xyw, xyz\}$ such that $u \notin B$. Since $r_v, r_w, r_z, r_x, r_{uv}, r_{uw}, r_{uz}$, and r_{ux} are even, we must have $v, w, z, x \in B$, which is impossible. Similarly $a \neq v, w, z$. If $a = x$ (or $a = y$), then we have $r_y = 8$ and consequently $r_u = r_{ux} + r_{uy} - 1$ which is not even. Therefore, $r_{xy} \leq 2$ and there are exactly 6 points x with $r_x = 4$. Now consider the six pairs appeared in the blocks containing a. At least two of them (like xy and xz) must appear in one of the four blocks which do not contain a. Note also that $r_x = 4$ and if we suppose that the fourth block containing x is xuv, then $r_{xu} = 1$ which is a contradiction. This completes the proof. $\square$

4 Existence results

In this section, we construct a $\mathrm{T}(s, v)$ for every possible volume s when v is even. Our method for $v \equiv 2 \pmod 4$ is similar to the construction of a $\mathrm{T}(M(v), v)$ trade in [1, 8]. The same approach in combination with a construction method for a $\mathrm{T}(M(v), v)$ trade in [11] is used for the case $v \equiv 0 \pmod 4$.

Lemma 7 *There is a $T(s, 10)$ if and only if $s = 8, 9, \ldots, 54, 56, 60$.*

Proof If there is a $\mathrm{T}(s, 10)$, then by Lemmas 2 and 6, $s = 8, 9, \ldots, 54, 56, 60$. Now we establish the converse. Let $X = \{x_1, \ldots, x_8\}$ and $x, y \notin X$. Let $T_1 = \{\{x\}, \{y\}\}$ and let T_2 be a $\mathrm{T}(1, 2, 8)$ trade on X of volume s_2. Then $T_1 * T_2$ is a $\mathrm{T}(2, 3, 10)$ trade of volume $2s_2$. By Lemma 4, we have $2s_2 = 4, 6, 8, \ldots, 24$. Let T_3 be a $\mathrm{T}(s_3, 8)$ trade on X, where $s_3 = 6, 7, \ldots, 22, 24$ by Lemma 3. $T_1 * T_2$ and T_3 have no common blocks and so their union gives a $\mathrm{T}(s, 10)$ in which $s = 2s_2 + s_3$. By Lemma 1, we obtain that $s = 10, 11, \ldots, 46, 48$. A $\mathrm{T}(8, 10)$ is found from two block disjoint $\mathrm{T}(4, 6)$. A $\mathrm{T}(9, 10)$ exists by Lemma 2. For $s = 50, 52, 54, 56, 60$, a $\mathrm{T}(s, 10)$ can be obtained by the construction given in Lemma 8. Finally, via a simple computer program we find $\mathrm{T}(47, 10)$, $\mathrm{T}(51, 10)$ and $\mathrm{T}(53, 10)$ (not presented here). $\square$

Lemma 8 *Let $v \equiv 2 \pmod 4$. If $m(v) \leq s \leq M(v)$ and $s \neq m(v) + 1, M(v) - 5, M(v) - 3, M(v) - 2, M(v) - 1$, then there is a $T(s, v)$.*

Proof We proceed by induction on v, knowing that the assertion holds for $v = 6, 10$ from Lemmas 3 and 7. Let $v > 10$, $X = \{x_1, \ldots, x_v\}$, and $X_i = \{x_1, \ldots, x_i\}$ for $1 \leq i \leq v$. We define trades T_i of volume s_i for $1 \leq i \leq 6$ as follows: Let $T_1 = \{\{x_1, x_2\}, \{x_3, x_4\}\}$ and T_2 be a $\mathrm{T}(1, 2, v-5)$ trade on $X \setminus X_5$. Then $T_1 * T_2$ is a $\mathrm{T}(2, 3, v-1)$ trade of volume $4s_2$. Let T_3 be a $\mathrm{T}(1, 2, 5)$ trade of volume 5 on X_5 and T_4 be a $\mathrm{T}(0, 1, v-6)$ trade on $X \setminus X_6$. Then $T_3 * T_4$ is a $\mathrm{T}(2, 3, v-1)$ trade of volume $10s_4$. Finally, let T_5 be a $\mathrm{T}(2, 3, 6)$ trade on X_6 and T_6 be a $\mathrm{T}(s_6, v-4)$ trade on $X \setminus X_4$. $T_1 * T_2$, $T_3 * T_4$, T_5, and T_6 have no common blocks and so their union gives a $\mathrm{T}(s, v)$ in which $s = 4s_2 + 10s_4 + s_5 + s_6$. By Lemma 4 and the hypothesis of induction, we have

$$\begin{aligned}
s_2 &= 0, 2, 3, \ldots, S(v-5)-2, S(v-5);\\
4s_2 &= 0, 8, 12, \ldots, 4S(v-5)-8, 4S(v-5);\\
s_4 &= 0, 1, \ldots, (v-6)/2;\\
10s_4 &= 0, 10, \ldots, 5(v-6);\\
s_5 &= 4, 6, 10;\\
s_6 &= m(v-4), m(v-4)+2, m(v-4)+3, \ldots, M(v-4)-6, M(v-4)-4, M(v-4).
\end{aligned}$$

Hence, by Lemma 1, we obtain

$$s = m(v-4)+4, m(v-4)+6, m(v-4)+7, \ldots, M(v)-6, M(v)-4, M(v).$$

Now suppose that $m(v) \le s \le m(v-4)+5$ and $s \ne m(v)+1, m(v-4)+4$. If $v \equiv 10 \pmod{12}$, then $m(v) = m(v-4)+4$ and there is nothing left. Otherwise, we need to find $\mathrm{T}(m(v), v)$ and $\mathrm{T}(m(v)+3, v)$. Using suitable combinations of trades in Lemma 3, $\mathrm{T}(10,14), \mathrm{T}(13,14), \mathrm{T}(12,18)$ and $\mathrm{T}(15,18)$ are easily constructible. For $v > 18$, noting that $m(v) = m(v-12)+8$, $\mathrm{T}(m(v-12)+8, v)$ and $\mathrm{T}(m(v-12)+11, v)$ are constructed from $\mathrm{T}(m(v-12), v-12)$ and $\mathrm{T}(m(v-12)+3, v-12)$, respectively and $\mathrm{T}(8,12)$ (see Lemma 3). $\square$

Lemma 9 *Let $v \equiv 0 \pmod 4$. If $m(v) \le s \le M(v)$ and $s \ne m(v)+1, M(v)-1$, then there is a $T(s,v)$.*

Proof We proceed by induction on v. The assertion holds for $v = 8$ by Lemma 3. Let $v > 8$, $X = \{x_1, \ldots, x_v\}$ and $X_i = \{x_1, \ldots, x_i\}$ for $1 \le i \le v$. We define trades T_i of volume s_i for $1 \le i \le 6$ as follows: Let $T_1 = \{\{x_1, x_2\}, \{x_3, x_4\}\}$ and T_2 be a $\mathrm{T}(1,2,v-7)$ trade on $X \setminus X_7$. Then $T_1 * T_2$ is a $\mathrm{T}(2,3,v-3)$ trade of volume $4s_2$. Let T_3 be a $\mathrm{T}(1,2,7)$ trade of volume 9 on X_7 such that $x_5x_6, x_6x_7, x_7x_5 \notin T_3$ and let T_4 be a $\mathrm{T}(0,1,v-8)$ trade on $X \setminus X_8$. Then $T_3 * T_4$ is a $\mathrm{T}(2,3,v-1)$ trade of volume $18s_4$. Finally, let T_5 be a $\mathrm{T}(2,3,8)$ trade on X_8 (not necessarily with foundation size 8) such that $(P_3(X_4) \cup P_3(X_8 \setminus X_4)) \cap (T_5^+ \cup T_5^-) = \varnothing$ (see [11]) and let T_6 be a $\mathrm{T}(s_6, v-4)$ trade on $X \setminus X_4$. $T_1 * T_2$, $T_3 * T_4$, T_5 and T_6 have no common blocks and so their union gives a $\mathrm{T}(s,v)$ in which $s = 4s_2 + 18s_4 + s_5 + s_6$. By Lemma 4 and the hypothesis of induction, we have

$$\begin{aligned}
s_2 &= 0, 2, 3, \ldots, S(v-7)-2, S(v-7);\\
4s_2 &= 0, 8, 12, \ldots, 4S(v-7)-8, 4S(v-7);\\
s_4 &= 0, 1, \ldots, (v-8)/2;\\
18s_4 &= 0, 18, \ldots, 9(v-8);\\
s_5 &= 4, 24;\\
s_6 &= m(v-4), m(v-4)+2, m(v-4)+3, \ldots, M(v-4)-2, M(v-4).
\end{aligned}$$

Now, by Lemma 1, we obtain

$$s = m(v-4)+4, m(v-4)+6, m(v-4)+7, \ldots, M(v)-2, M(v).$$

Now suppose that $m(v) \leq s \leq m(v-4)+5$ and $s \neq m(v)+1, m(v-4)+4$. If $v \equiv 4 \pmod{12}$, then $m(v) = m(v-4)+4$ and there is nothing left. Otherwise, we need to find $\mathrm{T}(m(v), v)$ and $\mathrm{T}(m(v)+3, v)$. Using suitable combinations of trades in Lemma 3, $\mathrm{T}(8,12)$ and $\mathrm{T}(11,12)$ are easily constructible. For $v \geq 20$, noting that $m(v) = m(v-12)+8$, $\mathrm{T}(m(v-12)+8, v)$ and $\mathrm{T}(m(v-12)+11, v)$ are constructed from $\mathrm{T}(m(v-12), v-12)$ and $\mathrm{T}(m(v-12)+3, v-12)$, respectively and $\mathrm{T}(8,12)$ (see Lemma 3). $\square$

We summarize the results in the following theorem.

Theorem 2 *Let $v \geq 6$ be even and let $m(v)$ and $M(v)$ be as defined in* (1) *and* (2)*.*

(i) If $v \equiv 0 \pmod 4$, then there is a $T(s, v)$ if and only if

$$s = m(v), m(v)+2, m(v)+3, \ldots, M(v)-2, M(v)$$

and possibly $s = m(v)+1$.

(ii) If $v \equiv 2 \pmod 4$, then there is a $T(s, v)$ if and only if

$$s = m(v), m(v)+2, m(v)+3, \ldots, M(v)-6, M(v)-4, M(v)$$

and possibly $s = m(v)+1$.

(iii) There is a $T(m(v)+1, v)$ if and only if $v \equiv 2, 4 \pmod 6$.

Note Using the technique of products of trades described here and some messy arguments, we have been able to determine the spectrum of trades with odd foundation size v not congruent to 5 modulo 6. For $v \equiv 5 \pmod 6$, about $v/6$ of volumes are undecided. We are hopeful that a better method to tackle the remaining parts could be found in a not distanced future.

References

[1] S. Ajoodani-Namini and G.B. Khosrovshahi, *More on halving the complete desings,* Discrete Math. **135** (1994), 29–37.

[2] E. J. Billington, *Combinatorial trades: a survey of recent results,* in Designs 2002: further combinatorial and constructive design theory (edited W.D. Wallis), Kluwer Academic Press, Norwell, Massachusetts, USA, 2002.

[3] D. E. Bryant, *On the volume of trades in triple systems,* Australas. J. Combin. **15** (1997), 161–176.

[4] R. H. F. Denniston, *Sylvester's problem of the* 15 *schoolgirls*, Discrete Math. **9** (1974), 229–233.

[5] Z. Eslami, G. B. Khosrovshahi and B. Tayfeh-Rezaie, *On Classification of* 2-$(8,3)$ *and* 2-$(9,3)$ *trades,* J. Combin. Math. Combin. Comput. **38** (2001), 231–242.

[6] Z. Eslami and B. Tayfeh-Rezaie, *On* 2-$(v,3)$ *trades of minimum volume*, Australas. J. Combin. **19** (1999), 239–251.

[7] Brenton D. Gray and Colin Ramsay, *On the spectrum of* $[v,k,t]$ *trades*, J. Statist. Plann. Inference **69** (1998), 1–19.

[8] G. B. Khosrovshahi and S. Ajoodani-Namini, *Combining* t*-designs,* J. Combin. Theory Ser. A **58** (1991), 26–34.

[9] G. B. Khosrovshahi, H. R. Maimani and R. Torabi, *On trades: an update,* Proceedings of the Conference on Optimal Discrete Structures and Algorithms—ODSA '97 (Rostock), Discrete Appl. Math. **95** (1999), 361–376.

[10] G. B. Khosrovshahi, H. R. Maimani and R. Torabi, *Classification of* 2-$(6,3)$ *and* 2-$(7,3)$ *trades*, Australas. J. Combin. **19** (1999), 55–72.

[11] G. B. Khosrovshahi and R. Torabi, *Maximal trades*, Ars Combin. **51** (1999), 211–223.