On the spectrum of simple $\mathrm{T}(2,3,v)$ trades[1]

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Abstract

In this paper, we determine the spectrum (the set of all possible volumes) of simple $\mathrm{T}(2,3,v)$ trades for any even foundation size v.

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1 Introduction

Let v, k and t be integers such that $v \geq k \geq t \geq 0$ and let X be a v-set. Let $P_k(X)$ denote the set of all k-subsets of X. A t-(v, k, λ) *design* (briefly t-design) is a pair $(X, \mathcal{B})$ in which $\mathcal{B}$ is a collection of elements of $P_k(X)$ such that every t-subset of X appears in exactly λ elements of $\mathcal{B}$. In the literature, 2-designs denoted by BIBD (Balanced Incomplete Block Design) are well known.

A $T(t, k, v)$ *trade* T on X is a triple (X, T^+, T^-) such that T^+ and T^- are two disjoint collections of elements (called *blocks*) of $P_k(X)$ such that for every t-subset A of X, the number of occurrences of A in the blocks of T^+ and of T^- are equal. For the sake of brevity, we omit the underlying set X from the notation and adopt the more convenient notation $T = \{T^+, T^-\}$. The *foundation* of T is the set of elements of X covered by T^+ (T^-) and the number of blocks in T^+ (T^-) is called the *volume* of T. If there is no repeated block in T^+ (T^-), then T is said to be *simple*. In this paper, we are only concerned with simple trades noting that trades with

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repeated blocks are also of some interest. T is called a *Steiner* trade if the occurrence of every t-subset of X in the blocks of T^+ (T^-) is at most once.

The family of t-designs are among the most important and fundamental families studied in combinatorial design theory. The t-designs are useful in constructing other classes of combinatorial designs. They also show up in other areas of combinatorics such as graph theory, coding theory, group theory, finite geometry and so on. In particular, BIB designs enjoy statistical optimality properties. Closely related to t-designs are trades which provide an algebraic setting for the study of t-designs. In principle, it is possible to produce all nonisomorphic t-designs from trades. They are also employed to construct designs with repeated blocks which also have applications in design of experiments. For some applications of trades, the reader can consult for example [6, 10, 11, 12, 20, 23]). The notion of trades first appeared in the context of block designs, but in recent years other types of trades such as Latin trades (for Latin squares) and graphical trades (for graphs) have also been introduced. For a survey of the subject, the reader is referred to [2, 18] and the references therein.

The determination of the set of all possible volumes of $\mathrm{T}(t,k,v)$ trades, the *spectrum*, has been studied by different authors. The spectrum of a trade can be considered for a fixed (varying) foundation size of trades. Trades could also be simple or with repeated blocks. Steiner trades have also attracted a considerable attention. A list of references includes [3, 7, 8, 9, 13, 14, 16, 21, 22]. The spectrum of $\mathrm{T}(2,3,v)$ trades with repeated blocks has been dealt with in [9]. The aim of this paper is to consider the spectrum of simple $\mathrm{T}(2,3,v)$ trades with a given foundation size. The spectrum of Steiner $\mathrm{T}(2,3,v)$ trades of a fixed foundation size is given in [3]. For a given foundation size v, the spectrum of Steiner and simple $\mathrm{T}(2,3,v)$ trades is roughly of size $v^2/6$ and $v^3/12$, respectively. Therefore, a large part of the spectrum for simple trades remains unattained. In this paper we determine the spectrum of simple $\mathrm{T}(2,3,v)$ trades for every even foundation size v. The main tool in achieving this is a "product of trades". To be more specific, we first determine the spectrum of $\mathrm{T}(1,2,v)$ trades. Then these trades are extended to $\mathrm{T}(2,3,v)$ trades via an extension theorem which is provided by the notion of product of trades.

It is easy to see that a $\mathrm{T}(t,k,v)$ trade is at the same time a $\mathrm{T}(t',k,v)$ trade for $0 \le t' < t$. This is an elementary result in the theory of t-designs. On the other hand, an important problem which seems to be more complicated is to determine when a $\mathrm{T}(t,k,v)$ trade can be obtained from given $\mathrm{T}(t',k',v')$ trades, where $t' < t$. To give an answer to this question, we introduce product of trades. The following definition is a special case of a general notion which was introduced in [1]. Let X_1 and X_2 be two disjoint sets of cardinalities v_1 and v_2, respectively. For $\mathcal{B}_1 \subseteq P_{k_1}(X_1)$ and $\mathcal{B}_2 \subseteq P_{k_2}(X_2)$, we let

$$\mathcal{B}_1 * \mathcal{B}_2 = \{A_1 \cup A_2 | A_1 \in \mathcal{B}_1, \quad A_2 \in \mathcal{B}_2\}.$$

Let $T_i = \{T_i^+, T_i^-\}$ be a $\mathrm{T}(t_i,k_i,v_i)$ trade of volume s_i on X_i for $i = 1,2$. Then, the *product* of

T_1 and T_2, denoted by $T_1 * T_2$, is defined as

$$T_1 * T_2 = \{(T_1^+ * T_2^+) \cup (T_1^- * T_2^-), (T_1^+ * T_2^-) \cup (T_1^- * T_2^+)\}.$$

The following is an extension theorem concerning the product of trades.

Theorem 1 [1] *$T_1 * T_2$ is a $\mathrm{T}(t_1 + t_2 + 1, k_1 + k_2, v_1 + v_2)$ trade of volume $2s_1s_2$.*

We also need to define the union operation of trades. Let $T_i = \{T_i^+, T_i^-\}$ be $\mathrm{T}(t, k, v_i)$ trades for $i = 1, 2$ with disjoint block sets, i.e. $T_1^+ \cap T_1^- \cap T_2^+ \cap T_2^- = \varnothing$. Then, the *union* of T_1 and T_2 is a $\mathrm{T}(t, k, v_1 + v_2)$ trade defined as $\{T_1^+ \cup T_2^+, T_1^- \cup T_2^-\}$.

Finally, we present some known results on the structure of trades. It is well known that in a $\mathrm{T}(t, k, v)$ trade the cardinality of foundation is at least $t + k + 1$ and the volume is at least 2^t [15]. Trades of volume 2^t have a unique structure and are called *basic* trades. For example the unique $\mathrm{T}(2, 3, 6)$ trade T of volume 4 is as follows:

$$\begin{aligned} T^+ &= \{xab, xcd, yac, ybd\}, \\ T^+ &= \{yab, ycd, xac, xbd\}, \end{aligned}$$

where xab refers to $\{x, a, b\}$ and so on. Note that $\mathrm{T}(0, k, v)$ trades are just two disjoint sets (of k-subsets of a v-set) of the same cardinality. $\mathrm{T}(1, 2, v)$ trades are exactly two edge disjoint graphs on v points such that the degree of each point is the same for both graphs. Trades with $t > 1$ have more complicated structures.

2 Preliminaries

In this section we present the known results on the spectrum of $\mathrm{T}(2, 3, v)$ trade. Hereafter, we denote a simple $\mathrm{T}(2, 3, v)$ trade of foundation size u and volume s by $\mathrm{TS}(s, u)$. By the notation $s = a_1, a_2, \ldots, a_r$, we mean that s may take any of the values $a_1, a_2, \ldots, a_r$. The following lemma will be used in the next sections frequently.

Lemma 1 *Let $a_1, \ldots, a_r, b, c$ be integers such that $a_i \geq a_{i-1}$ and $c - b \geq a_i - a_{i-1} - 1$ for $2 \leq i \leq r$. If $s_1 = a_1, \ldots, a_r$ and $s_2 = b, b+1, \ldots, c$, then $s_1 + s_2 = a_1 + b, a_1 + b + 1, a_1 + b + 2, \ldots, a_r + c$.*

Proof Let $s_1 = a_1, \ldots, a_r$ and $s_2 = b, b + 1, \ldots, c$. We want to show that $s_1 + s_2$ takes any value $a_1 + b \leq h \leq a_r + c$. For any $1 \leq i \leq r$, $s_1 + s_2$ takes any value $a_i + b \leq h \leq a_i + c$. Since $c - b \geq a_i - a_{i-1} - 1$ (for $2 \leq i \leq r$), we have $a_{i-1} + c + 1 \geq a_i + b$ and therefore there will be no gap for $s_1 + s_2$ in the interval $[a_1 + b, a_r + c]$. □

Let $v \geq 6$ be even. We define $m(v)$ and $M(v)$ as

$$m(v) = 2\lceil \frac{v}{3} \rceil, \tag{1}$$

and

$$M(v) = \begin{cases} \frac{v(v-1)(v-2)}{12} & \text{if } v \equiv 2 \pmod 4, \\ \frac{v(v+1)(v-4)}{12} & \text{if } v \equiv 0 \pmod 4. \end{cases} \tag{2}$$

We summarize some of the known results about the spectrum of $\mathrm{T}(2,3,v)$ trades in the following lemmas.

Lemma 2 **[3, 5, 21]** *If there is a* $\mathrm{TS}(s,v)$*, then* $m(v) \leq s \leq M(v)$*. Moreover, there is a* $\mathrm{TS}(m(v)+1, v)$ *if and only if* $v \not\equiv 0 \pmod 6$*.*

Lemma 3 **[4, 19]** *Let* $6 \leq v \leq 8$*. Then, there is a* $\mathrm{TS}(s,v)$ *if and only if one of the following occurs:*

(i) $v = 6, \quad s = 4, 6, 10;$

(ii) $v = 7, \quad s = 6, 7, 9, 10, 12;$

(iii) $v = 8, \quad s = 6, 7, \ldots, 22, 24.$

Theorem 1 suggests that $\mathrm{T}(2,3,v)$ trades may be constructed from $\mathrm{T}(1,2,v)$ trades. Therefore, it is natural to determine the possible volumes for $\mathrm{T}(1,2,v)$ trades. The results will be used in the subsequent sections. For $v \geq 4$, let

$$S(v) = \begin{cases} \frac{v(v-2)}{4} & \text{if } v \equiv 0 \pmod 2, \\ \frac{v(v-1)}{4} & \text{if } v \equiv 1 \pmod 4, \\ \frac{(v+2)(v-3)}{4} & \text{if } v \equiv 3 \pmod 4. \end{cases}$$

Lemma 4 *Let* $v \geq 4$*. Then there is a* $\mathrm{T}(1,2,v)$ *trade of volume* s *if and only if* $2 \leq s \leq S(v)$*, and* $s \neq S(v) - 1$ *in case* $v \equiv 1 \pmod 4$*.*

Proof Suppose that $T = \{T^+, T^-\}$ is a $\mathrm{T}(1,2,v)$ trade of volume s. Then $s \leq \binom{v}{2}/2 = v(v-1)/4$. Let v be even. Then the number of occurrences of every point in $T^+ \cup T^-$ must be even. Therefore the total number of occurrences of points in $T^+ \cup T^-$ is at most $v(v-2)$ and so $s \leq \binom{v}{2}/2 = v(v-1)/4$. Next let $v \equiv 3 \pmod 4$. Then then the number of blocks in $T^+ \cup T^-$ which is even cannot be $v(v-1)/2$. If it is $v(v-1)/2 - 1$, then there is a unique 2-subset $\{x, y\}$ which is not a block of T. But then the number of occurrences of x in $T^+ \cup T^-$ is $v - 2$, an odd number, a contradiction. Hence $s < (v(v-1)/2-3)/2 = (v+2)(v-3)/4$. Finally, let $v \equiv 1 \pmod 4$ and $s = S(v)-1$. Then there is a point which appears $v-2$ times in $T^+ \cup T^-$, a contradiction since $v-2$ is odd.

The proof of existence is by induction on v. The assertion can easily be checked for $v = 4, 5$. Therefore suppose that $v \geq 6$. Let $X = \{x_1, \ldots, x_v\}$ and $X_i = \{x_1, \ldots, x_i\}$ for $1 \leq i \leq v$. First suppose that $v \equiv 0 \pmod 2$ or $v \equiv 3 \pmod 4$. Let T_1 be a $\mathrm{T}(0,1,2)$ trade with $T_1^+ = \{x_1\}$ and $T_1^- = \{x_2\}$ and let T_2 be a $\mathrm{T}(0,1,v-2)$ trade of volume s_2 on $X \setminus X_2$. Then by Theorem 1, $T_1 * T_2$ is a $\mathrm{T}(1,2,v)$ trade of volume $2s_2$. Let T_3 be a $\mathrm{T}(1,2,v-2)$ trade of volume s_3 on $X \setminus X_2$. Then the union of $T_1 * T_2$ and T_3 gives a $\mathrm{T}(1,2,v)$ trade of volume $s = 2s_2 + s_3$. If $v \equiv 0 \pmod 2$, then $s_2 = 0, 1, \ldots, (v-2)/2$, $s_3 = 2, 3, \ldots, S(v-2)$ (by the induction hypothesis) and so by Lemma 1, we obtain $s = 2, 3, \ldots, S(v)$. Now assume that $v \equiv 3 \pmod 4$. We have $s_2 = 0, 1, \ldots, (v-3)/2$ and $s_3 = 2, 3, \ldots, S(v-2)-2, S(v-2)$ (by the induction hypothesis). By Lemma 1, we obtain $s = 2, 3, \ldots, S(v)-2, S(v)$. A trade of volume $S(v)-1$ can easily be constructed from a trade T of volume $S(v)$ constructed as above. Without loss of generality, we may assume that $x_1x_3, x_2x_4 \in T^+$ and $x_3x_4 \in T^-$. We remove x_1x_3 and x_2x_4 from T^+ and x_3x_4 from T^- and add x_1x_2 to T^+. Then we find a trade of the desired volume.

The case $v \equiv 1 \pmod 4$ is dealt with in a similar way. Let T_1 be a $\mathrm{T}(0,1,4)$ trade with $T_1^+ = \{x_1, x_2\}$ and $T_1^- = \{x_3, x_4\}$ and let T_2 be a $\mathrm{T}(0,1,v-5)$ trade of volume s_2 on $X \setminus X_5$. Then $T_1 * T_2$ is a $\mathrm{T}(1,2,v-1)$ trade of volume $4s_2$. Let T_3 and T_4 be $\mathrm{T}(1,2,5)$ and $\mathrm{T}(1,2,v-4)$ trades of volume s_3 and s_4 on X_5 and $X \setminus X_4$, respectively. Then the union of $T_1 * T_2$, T_3, and T_4 gives a $\mathrm{T}(1,2,v)$ trade of volume $s = 4s_2 + s_3 + s_4$. We have $s_2 = 0, 1, \ldots, (v-5)/2, s_3 = 0, 2, 3, 5$ and $s_4 = 2, 3, \ldots, S(v-4)-2, S(v-4)$ (by the induction hypothesis). Hence, by Lemma 1, we obtain $s = 2, 3, \ldots, S(v)-2, S(v)$, as required. $\square$

3 Nonexistence results

In this section, we show that some special volumes do not occur in the spectrum of $\mathrm{T}(2,3,v)$ trades.

Lemma 5 *If $v \equiv 0 \pmod 4$, then there is no* $\mathrm{TS}(M(v)-1, v)$.

Proof Let X be a v-set and suppose that $T = \{T^+, T^-\}$ is a $\mathrm{TS}(M(v)-1, v)$ trade on X. Let $S = P_3(X) \setminus (T^+ \cup T^-)$. Then $|S| = v+2$ by (2). The number of occurrences of each pair of distinct points $x, y \in X$ in $P_3(X)$ is $v-2$ which is even and hence the number of occurrences of xy in S is even. Let r_x be the number of occurrences of $x \in X$ in S. The number of occurrences of x in $P_3(X)$ is $\binom{v-1}{2}$ which is odd and hence r_x is odd and so at least 3 by the above argument.

Therefore, there are at most three points x with $r_x > 3$. Let a be a point such that $r_a = 3$. Let $\{abc, abd, acd\} \subseteq S$ for some $b, c, d \in X$. If (for example) $r_b = 3$, then $bcd \in S$. We collect all the triples of this type in S'. Then $S \setminus S'$ must contain 6 triples on 4 points (say x, y, z, t) such that $r_x = 3$ and $r_y = r_z = r_t = 5$. But then the six distinct triples must be $xyz, xyt, xzt, yzt, yzt, yzt$, a contradiction. $\square$

Lemma 6 *If $v \equiv 2 \pmod 4$, then there is no* $\mathrm{TS}(s, v)$ *for $s = M(v) - 5, M(v) - 3, M(v) - 2, M(v) - 1$.*

Proof Let X be a v-set and suppose that $T = \{T^+, T^-\}$ is a $\mathrm{TS}(s, v)$ trade on X with $s \in \{M(v)-5, M(v)-3, M(v)-2, M(v)-1\}$. Let $S = P_3(X) \setminus (T^+ \cup T^-)$. Then $|S| \in \{2, 4, 6, 10\}$. Let r_x and r_{xy} be the numbers of occurrences of point x and pair xy in S, respectively. The number of occurrences of x in $P_3(X)$ is $\binom{v-1}{2}$ which is even and hence r_x is even. The number of occurrences of each pair of distinct points x, y in $P_3(X)$ is $v - 2$ which is even and hence r_{xy} is even. This also implies $r_x \neq 2$. Clearly, the number of points x such that $r_x \neq 0$ is at least 5. It yields that $|S| > 6$ and so $|S| = 10$. Since $\sum_{x \in X} r_x = 30$, there must be a point a such that r_a is not divisible by 4 and therefore $r_a = 6$ or 10. But $r_a = 10$ is impossible since r_{xy} is even for any pair xy. So $r_a = 6$. Now we show that for any pair xy, $r_{xy} \leq 2$. It is clear that $r_{xy} \leq 4$, since otherwise there are at least 8 points x with $r_x \neq 0$ and hence $\sum_{x \in X} r_x \geq 32$, a contradiction. Suppose that $r_{xy} = 4$ and $xyu, xyv, xyw, xyz \in S$. We have $r_x = r_y = 6$ and without loss of generality $r_u = 6$. The number of triples in S containing x or y is 8 and these triples contain u exactly 3 times and so $r_x \leq 5$, a contradiction. Therefore, $r_{xy} \leq 2$ and there are exactly 6 points p with $r_p = 4$. Now consider the six pairs that appear in the blocks containing a, but do not themselves contain a. At least two of them (say xy and xz) must appear in one of the four blocks which do not contain a. Note also that $r_x = 4$ and if we suppose that the fourth block containing x is xuv, then $r_{xu} = 1$ which is a contradiction. This completes the proof. $\square$

4 Existence results

In this section, we construct a $\mathrm{TS}(s, v)$ for every possible volume s when v is even. Our method for $v \equiv 2 \pmod 4$ is similar to the construction of a $\mathrm{TS}(M(v), v)$ trade in [1, 17]. The same approach in combination with a construction method for a $\mathrm{TS}(M(v), v)$ trade in [21] is used for the case $v \equiv 0 \pmod 4$.

Lemma 7 *There is a* $\mathrm{TS}(s, 10)$ *if and only if $s = 8, 9, \ldots, 54, 56, 60$.*

Proof If there is a $\mathrm{TS}(s, 10)$, then by Lemmas 2 and 6, $s = 8, 9, \ldots, 54, 56, 60$. Now we establish the converse. Let $X = \{x_1, \ldots, x_8\}$ and $x, y \notin X$. Let T_1 be a $\mathrm{T}(0, 1, 2)$ trade with

$T_1^+ = \{x\}$ and $T_1^- = \{y\}$ and let T_2 be a $\mathrm{T}(1,2,8)$ trade on X of volume s_2. Then $T_1 * T_2$ is a $\mathrm{T}(2,3,10)$ trade of volume $2s_2$. By Lemma 4, we have $2s_2 = 4, 6, 8, \ldots, 24$. Let T_3 be a $\mathrm{TS}(s_3, 8)$ trade on X, where $s_3 = 6, 7, \ldots, 22, 24$ by Lemma 3. $T_1 * T_2$ and T_3 have no common blocks and so their union gives a $\mathrm{TS}(s, 10)$ in which $s = 2s_2 + s_3$. By Lemma 1, we obtain that $s = 10, 11, \ldots, 46, 48$. A $\mathrm{TS}(8, 10)$ is found from two block disjoint $\mathrm{TS}(4, 6)$ as follows: Let $T = \{\{acd, aef, bce, bdf\}, \{ace, adf, bcd, bef\}\}$ and $T' = \{\{axy, azt, bxz, byt\}, \{axz, ayt, bxy, bzt\}\}$. T and T' are $\mathrm{TS}(4, 6)$ trades and their union is a $\mathrm{TS}(8, 10)$ trade. A $\mathrm{TS}(9, 10)$ exists by Lemma 2. For $s = 50, 52, 54, 56, 60$, a $\mathrm{TS}(s, 10)$ can be obtained by the construction given in Lemma 8. Finally, via a simple computer program we find $\mathrm{TS}(47, 10)$, $\mathrm{TS}(51, 10)$ and $\mathrm{TS}(53, 10)$ which are accessible through the web page http://math.ipm.ac.ir/tayfeh-r/research/trades10.htm. $\square$

Lemma 8 *Let* $v \equiv 2 \pmod 4$. *If* $m(v) \le s \le M(v)$ *and* $s \ne m(v)+1, M(v)-5, M(v)-3, M(v)-2, M(v)-1$, *then there exists a* $\mathrm{TS}(s, v)$.

Proof We proceed by induction on v, knowing that the assertion holds for $v = 6, 10$ from Lemmas 3 and 7. Let $v > 10$, $X = \{x_1, \ldots, x_v\}$ and $X_i = \{x_1, \ldots, x_i\}$ for $1 \le i \le v$. We define trades T_i for $1 \le i \le 6$ as follows: Let T_1 be a $\mathrm{T}(0,1,4)$ trade with $T_1^+ = \{x_1, x_2\}$ and $T_1^- = \{x_3, x_4\}$ and T_2 be a $\mathrm{T}(1,2,v-5)$ trade on $X \setminus X_5$. Let T_3 be a $\mathrm{T}(1,2,5)$ trade of volume 5 on X_5 and T_4 be a $\mathrm{T}(0,1,v-6)$ trade on $X \setminus X_6$. Finally, let T_5 be a $\mathrm{T}(2,3,6)$ trade on X_6 and T_6 be a $\mathrm{TS}(s_6, v-4)$ trade on $X \setminus X_4$. Let s_i denote the volume of T_i for $1 \le i \le 6$. Then $T_1 * T_2$ is a $\mathrm{T}(2,3,v-1)$ trade of volume $4s_2$ and $T_3 * T_4$ is a $\mathrm{T}(2,3,v-1)$ trade of volume $10s_4$. $T_1 * T_2$, $T_3 * T_4$, T_5, and T_6 have no common blocks and so their union gives a $\mathrm{TS}(s, v)$ in which $s = 4s_2 + 10s_4 + s_5 + s_6$. By Lemma 4 and the hypothesis of induction, we have

$$\begin{aligned}
s_2 &= 0, 2, 3, \ldots, S(v-5)-2, S(v-5);\\
4s_2 &= 0, 8, 12, \ldots, 4S(v-5)-8, 4S(v-5);\\
s_4 &= 0, 1, \ldots, (v-6)/2;\\
10s_4 &= 0, 10, \ldots, 5(v-6);\\
s_5 &= 4, 6, 10;\\
s_6 &= m(v-4), m(v-4)+2, m(v-4)+3, \ldots, M(v-4)-6, M(v-4)-4, M(v-4).
\end{aligned}$$

Hence, by Lemma 1, we obtain

$$s = m(v-4)+4, m(v-4)+6, m(v-4)+7, \ldots, M(v)-6, M(v)-4, M(v).$$

Now suppose that $m(v) \le s \le m(v-4)+5$ and $s \ne m(v)+1, m(v-4)+4$. If $v \equiv 10 \pmod{12}$, then $m(v) = m(v-4)+4$ and there is nothing left to prove. Let $v \equiv 2, 6 \pmod{12}$. Then $m(v) = m(v-4)+2$ and we need to find $\mathrm{TS}(m(v), v)$ and $\mathrm{TS}(m(v)+3, v)$. Using suitable unions of trades in Lemmas 3 and 7, $\mathrm{TS}(10,14), \mathrm{TS}(13,14), \mathrm{TS}(12,18)$ and $\mathrm{TS}(15,18)$ are easily constructible. For $v > 18$, noting that $m(v) = m(v-12)+8$, a $\mathrm{TS}(m(v-12)+8, v)$ is obtained from the union of a $\mathrm{TS}(m(v-12), v-12)$ and a block disjoint $\mathrm{TS}(8,12)$ trade (see Lemma 3) and similarly a $\mathrm{TS}(m(v-12)+11, v)$ is constructed from a $\mathrm{TS}(m(v-12)+3, v-12)$ and a block disjoint $\mathrm{TS}(8,12)$ trade. $\square$

Lemma 9 *Let $v \equiv 0 \pmod 4$. If $m(v) \le s \le M(v)$ and $s \ne m(v)+1, M(v)-1$, then there exists a $\mathrm{TS}(s,v)$.*

Proof We proceed by induction on v. The assertion holds for $v = 8$ by Lemma 3. Let $v > 8$, $X = \{x_1, \dots, x_v\}$ and $X_i = \{x_1, \dots, x_i\}$ for $1 \le i \le v$. We define trades T_i for $1 \le i \le 6$ as follows: Let T_1 be a $\mathrm{T}(0,1,4)$ trade with $T_1^+ = \{x_1, x_2\}$ and $T_1^- = \{x_3, x_4\}$ and T_2 be a $\mathrm{T}(1,2,v-7)$ trade on $X \setminus X_7$. Let T_3 be a $\mathrm{T}(1,2,7)$ trade of volume 9 on X_7 such that $x_5x_6, x_6x_7, x_7x_5 \notin T_3$ and let T_4 be a $\mathrm{T}(0,1,v-8)$ trade on $X \setminus X_8$. Finally, let T_5 be a $\mathrm{T}(2,3,8)$ trade on X_8 (not necessarily with foundation size 8) such that $(P_3(X_4) \cup P_3(X_8 \setminus X_4)) \cap (T_5^+ \cup T_5^-) = \varnothing$ (see [21]) and let T_6 be a $\mathrm{TS}(s_6, v-4)$ trade on $X \setminus X_4$. Let s_i denote the volume of T_i for $1 \le i \le 6$. Then $T_1 * T_2$ is a $\mathrm{T}(2,3,v-3)$ trade of volume $4s_2$ and $T_3 * T_4$ is a $\mathrm{T}(2,3,v-1)$ trade of volume $18s_4$. $T_1 * T_2$, $T_3 * T_4$, T_5 and T_6 have no common blocks and so their union gives a $\mathrm{TS}(s,v)$ in which $s = 4s_2 + 18s_4 + s_5 + s_6$. By Lemma 4 and the hypothesis of induction, we have

$$\begin{aligned}
s_2 &= 0, 2, 3, \dots, S(v-7)-2, S(v-7);\\
4s_2 &= 0, 8, 12, \dots, 4S(v-7)-8, 4S(v-7);\\
s_4 &= 0, 1, \dots, (v-8)/2;\\
18s_4 &= 0, 18, \dots, 9(v-8);\\
s_5 &= 4, 24;\\
s_6 &= m(v-4), m(v-4)+2, m(v-4)+3, \dots, M(v-4)-2, M(v-4).
\end{aligned}$$

Now, by Lemma 1, we obtain

$$s = m(v-4)+4, m(v-4)+6, m(v-4)+7, \dots, M(v)-2, M(v).$$

Now suppose that $m(v) \le s \le m(v-4)+5$ and $s \ne m(v)+1, m(v-4)+4$. If $v \equiv 4 \pmod{12}$, then $m(v) = m(v-4)+4$ and there is nothing left to prove. Let $v \equiv 0, 8 \pmod{12}$. Then $m(v) = m(v-4)+2$ and we need to find $\mathrm{TS}(m(v), v)$ and $\mathrm{TS}(m(v)+3, v)$. Using suitable unions of trades in Lemma 3, $\mathrm{TS}(8,12)$ and $\mathrm{TS}(11,12)$ are easily constructible. For $v \ge 20$, noting that $m(v) = m(v-12)+8$, a $\mathrm{TS}(m(v-12)+8, v)$ is obtained from the union of a $\mathrm{TS}(m(v-12), v-12)$ and a block disjoint $\mathrm{TS}(8,12)$ trade (see Lemma 3) and similarly a $\mathrm{TS}(m(v-12)+11, v)$ is constructed from a $\mathrm{TS}(m(v-12)+3, v-12)$ and a block disjoint $\mathrm{TS}(8,12)$ trade. $\square$

We summarize the results in the following theorem.

Theorem 2 *Let $v \geq 6$ be even and let $m(v)$ and $M(v)$ be as defined in* (1) *and* (2)*.*

(i) Let $v \equiv 0 \pmod 4$. Then there exists a $\mathrm{TS}(s, v)$ *if and only if*

$$s = m(v), m(v) + 2, m(v) + 3, \dots, M(v) - 2, M(v)$$

and also $s = m(v) + 1$ when $v \equiv 4, 8 \pmod{12}$.

(ii) Let $v \equiv 2 \pmod 4$. Then there exists a $\mathrm{TS}(s, v)$ *if and only if*

$$s = m(v), m(v) + 2, m(v) + 3, \dots, M(v) - 6, M(v) - 4, M(v)$$

and also $s = m(v) + 1$ when $v \equiv 2, 10 \pmod{12}$.

Note Using the technique of product of trades described here and some messy arguments, we have been able to determine the spectrum of trades with odd foundation size v not congruent to 5 modulo 6. For $v \equiv 5 \pmod 6$, about $v/6$ of volumes (roughly between $\binom{v}{3}/2 - v/6$ and $\binom{v}{3}/2$) are undecided. We are hopeful that a better method to tackle the remaining parts could be found in a not too distant future.

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