Trades and t-designs

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Abstract

Trades, as combinatorial objects, possess interesting combinatorial and algebraic properties and play a considerable role in various areas of combinatorial designs. In this paper we focus on trades within the context of t-designs. A pedagogical review of the applications of trades in constructing halving t-designs is presented. We also consider (N,t)-partitionable sets as a generalization of trades. This generalized notion provides a powerful approach to the construction of large sets of t-designs. We review the main recursive constructions and theorems obtained by this approach. Finally, we discuss the linear algebraic representation of trades and present two applications.

1 Introduction

Let v,k,t and λ be integers such that $v \geq k \geq t \geq 0$ and $\lambda \geq 1$. Let X be a v-set and let $P_i(X)$ denote the set of all i-subsets of X for any i. A t-(v,k,λ) *design* (briefly t-*design*) is a pair $D = (X, \mathcal{B})$ in which $\mathcal{B}$ is a collection of elements of $P_k(X)$ such that every $A \in P_t(X)$ appears in exactly λ elements of $\mathcal{B}$. Let N be a natural number greater than 1. A *large set of* t-(v,k,λ) *designs of size* N, $LS[N](t,k,v)$, is a set of N disjoint t-(v,k,λ) designs $(X,\mathcal{B}_i)$ such that $\{\mathcal{B}_i |\ 1 \leq i \leq N\}$ is a partition of $P_k(X)$. A $T(t,k,v)$ *trade* is a pair $T = (X, \{T^+, T^-\})$ in which T^+ and T^- are two disjoint collections of elements of $P_k(X)$ such that for every $A \in P_t(X)$, the number of occurrences of A in T^+ is the same as the number of occurrences of A in T^-, i.e. T^+ and T^- are mutually balanced. Note that some elements of $P_t(X)$ may not appear in T^+ and T^-. For simplicity, we write $(X; T^+, T^-)$ instead of $(X, \{T^+, T^-\})$.

The family of t-designs are among the most important and fundamental families of combinatorial designs. They show up in different areas of combinatorics such as coding theory, group theory, finite geometry and so on. They are employed in construction of various combinatorial designs and configurations. In particular, 2-designs, for their statistical optimality properties, are widely used in design of experiments.

Trades are useful combinatorial objects with interesting combinatorial and algebraic properties. They appear in various contexts of combinatorial design theory. Trades were first introduced in the theory of t-designs by some authors under different names in the seventies of the last century (see [17] for details). In recent years Latin trades in connection to Latin squares and graphical trades related to graphical designs have been investigated. For a survey of trades in different contexts we refer the reader to [9]. The reference [22] is a survey specifically devoted to applications of trades in the theory t-designs.

In this paper, we consider trades within the context of t-designs. Trades are very useful in the study of t-designs with many applications. They are utilized in constructing signed t-designs, nonisomorphic t-designs and t-designs with repeated blocks. They are also employed to determine the spectrum of support sizes of t-designs with repeated blocks, block intersection numbers and defining sets of t-

designs. For a thorough discussion of these applications, see [11, 17, 18, 22] and the references therein.

In the last twenty years trades and their generalization, i.e. (N,t)-partitionable sets have been successfully used in the study of the existence problem of t-designs and large sets. This study has led to a powerful approach to the construction of large sets of t-designs. The method was developed in the nineties of the last century by Ajoodani-Namini and Khosrovshahi [5] through their work on Hartman's conjecture (halving conjecture) on the existence of halving designs. Since then, many existence results as well as recursive constructions have been obtained using this approach for t-designs in general and for halving designs and large sets of prime sizes in particular. Undoubtedly, the most outstanding result achieved by this method is the proof of halving conjecture for 2-designs by Ajoodani-Namini [1]. We here present an instructive review of the approach of (N,t)-partitionable sets. For simplicity of presentation, our treatment is mainly based on trades, i.e. $(2,t)$-partitionable sets. We demonstrate how one can use trades to find recursive constructions for halving designs or large sets of size 2. The general case of (N,t)-partitionable sets is briefly discussed and it is shown that most of the results for large sets of size 2 can easily be extended to large sets in general. Trades also provide an algebraic setting for the study of t-designs. We describe the linear algebraic representation of trades and present two applications of it.

2 t-Designs and large sets

Let $D=(X,\mathcal{B})$ be a t-(v,k,λ) design. The parameter t is called the *strength* of D by some authors. The elements of X are called *points* and the elements of $\mathcal{B}$ are called *blocks*. When the blocks of D are distinct, the design is *simple*. In this paper, we are mainly interested in simple designs. An easy counting argument shows that if there exists a t-(v,k,λ) design, then

$$\lambda_i=\lambda\frac{\binom{v-i}{t-i}}{\binom{k-i}{t-i}}\quad\text{for}\quad 0\le i\le t,$$

are integers. These are known as the *feasibility conditions*. From these conditions it easily follows that D is an i-(v,k,λ_i) design for any $0\le i\le t$. The pair $(X,P_k(X))$ is a t-$(v,k,\binom{v-t}{k-t})$ design and is called a *complete design*. The following observation is well known.

Lemma 2.1 *Let $0\le t\le k\le v$. If there exists a t-(v,k,λ) design which is not complete, then $t<k<v-t$.*

The main question concerning t-designs is the existence problem: For given integers v,k,t,λ such that $0\le t\le k\le v$, $\lambda\ge 1$ and satisfying the feasibility conditions, does there exist a t-(v,k,λ) design? Naturally, the existence problem is in general intractable. However, it has been dealt with and answered in some special cases. For a survey of known results, the interested reader may consult [21, 33] and the references therein.

For some times it was believed that there is no simple t-design for $t\ge 6$. Then in 1982 Magliveras and Leavitt constructed the first examples of simple 6-designs [32].

Later on, in 1986 Kreher and Radziszowski [29] found the smallest possible 6-design, i.e. a 6-(14,7,4) design. Finally in 1987, Teirlinck [36] attained the striking result that simple t-designs exist for all t. In [3], Ajoodani-Namini gave a new proof of this result using a completely different methodology, i.e. the method (N, t)-partitionable sets which is the subject of this review.

The approach of (N, t)-partitionable sets provides a method to tackle the existence problem of t-designs through the construction of large sets of t-designs. Based on this method, some strong recursive constructions for large sets are obtained. Using these constructions we are able to establish many interesting existence results on t-designs. The method will be explained in detail in the subsequent sections. Here, we review some basic facts on large sets. The following theorems are easy to prove but contain important information.

Theorem 2.2 [3, 27] *If there exists an LS$[N](t, k, v)$, then there exist LS$[N](t-i, k-j, v-l)$ for all $0 \leq j \leq l \leq i \leq t$.*

Theorem 2.3 *There exists an LS$[N](t, k, v)$ if and only if there exists an LS$[N](t, v-k, v)$*

It is well known that a set of necessary conditions for the existence of an LS$[N](t, k, v)$ is

$$N \Big| \binom{v-i}{k-i} \quad \text{for} \quad 0 \leq i \leq t. \tag{2.1}$$

These conditions are direct consequences of the feasibility conditions for t-designs. They can also be deduced as follows. If the large set LS$[N](t, k, v)$ exists, then for $0 \leq i \leq t$, the set $P_{k-i}(X \setminus \{1, \ldots, i\})$ is partitioned into N equal parts. Note that the necessary conditions (2.1) are not always sufficient. A historic example is the nonexistence of LS$[5](2, 3, 7)$. It is worth to note that by a celebrated result due to Baranyai, the necessary conditions (2.1) are sufficient for the existence of large sets LS$[N](1, k, v)$ [7, 10, 16].

Let N, t, k and v be integers such that $N > 1$ and $0 \leq t \leq k \leq v$. The quadruple $(N; t, k, v)$ satisfying (2.1) is called a *feasible quadruple*. It is possible to give a better description for feasible quadruples when N is a prime power. Let m and n be positive integers. We denote the remainder of division m by n by $(m)_n$.

Theorem 2.4 [27] *Let p be a prime, α a positive integer and $0 \leq t \leq k \leq v$. The quadruple $(p^\alpha; t, k, v)$ is feasible if and only if there exist distinct positive integers ℓ_i $(1 \leq i \leq \alpha)$ such that $t \leq (v)_{p^{\ell_i}} < (k)_{p^{\ell_i}}$.*

Using Theorem 2.4, we can easily determine all the feasible quadruples when $N = p$ is a prime:

$$(p; t, k, v) = (p; t, mp^z + r, np^z + s), \tag{2.2}$$

where $0 \leq t \leq s < r < p^z$ and $0 \leq m < n$. We can also assume that z is the smallest or the largest number with the properties above to be assured of the uniqueness of the representation (2.2). We present two examples to show the importance of Theorem 2.4.

Example 2.5 By Theorem 2.4, the quadruple $(5; 2, 4, 13)$ is feasible since $2 \leq (13)_5 < (4)_5$ and also the quadruple $(11; 2, 4, 13)$ is feasible since $2 \leq (13)_{11} < (4)_{11}$. Now from (2.1), it is clear that the quadruple $(55; 2, 4, 13)$ is feasible.

Example 2.6 What is the largest value of t for which the parameter set of an LS[13]$(t, 9, 18)$ is feasible? By Theorem 2.4, we must have $t \leq (18)_{13^\alpha} < (9)_{13^\alpha}$ and hence $\alpha = 1$ and $t_{\max} = 5$.

3 Trades

We start this section with two examples of trades.

Example 3.1 Let $X = \{1, 2, 3, 4, 5, 6\}$. Let

$$T_1^+ = \{135, 146, 236, 245\},$$
$$T_1^- = \{136, 145, 235, 246\}.$$

Then $T_1 = (X; T_1^+, T_1^-)$ is a $\mathrm{T}(2, 3, 6)$ trade. Note that by 123 we mean $\{1, 2, 3\}$, etc.

Example 3.2 Here is another example of a $\mathrm{T}(2, 3, 6)$ trade. Let

$$T_2^+ = \{123, 124, 156, 256, 345, 346\},$$
$$T_2^- = \{125, 126, 134, 234, 356, 456\}.$$

Then $T_2 = (X; T_2^+, T_2^-)$ is a $\mathrm{T}(2, 3, 6)$ trade.

Lemma 3.3 *A $\mathrm{T}(t, k, v)$ trade T is also a $\mathrm{T}(t', k, v)$ trade for any $0 \leq t' < t$.*

By letting $t' = 0$ in this lemma, it follows that the number of blocks in T^+ is the same as the number of blocks in T^- which is called the *volume* of T. By letting $t' = 1$, we also observe that the set of points covered by T^+ is exactly the same as the set of points covered by T^-. This set is called the *foundation* of T. We are now ready to state the following fundamental theorem of trades.

Theorem 3.4 **[19]** *A nontrivial $\mathrm{T}(t, k, v)$ trade has foundation size at least $k+t+1$ and volume at least 2^t.*

A trade of volume 0 is called the *trivial trade*. By the above theorem and what follows, there exists a nontrivial $\mathrm{T}(t, k, v)$ trade if and only if $t < k < v - t$. A $\mathrm{T}(t, k, v)$ trade of foundation size $k+t+1$ and volume 2^t has a unique structure and is called the *minimal* trade. For example, any minimal $\mathrm{T}(2, 3, 6)$ trade is isomorphic to trade T_1 of Example 3.1. We also note that a $\mathrm{T}(0, k, v)$ trade $T = (X; T^+, T^-)$ has a simple structure. In fact the only restriction on T to be a $\mathrm{T}(0, k, v)$ is that T^+ and T^- contain the same number of blocks. For $t = 1$, it is easy to construct trades. However, for $t \geq 2$ the construction does not seem to be so trivial. The problem becomes harder as t increases. Even for $t = 3$, there are many unsolved problems on the existence of $\mathrm{T}(t, k, v)$ trades. In order to construct trades for any t there is a method based on the notion of products of trades which is to be presented in Section 4.

By definition, the blocks of a trade do not have to be distinct. However, trades with distinct blocks (simple trades), are of greater importance. Trades T_1 and T_2 of Examples 3.1 and 3.2, respectively, are examples of simple trades. There is also a trade on $X = \{1, 2, \ldots, 6\}$ which in not simple as the following example illustrates.

Example 3.5 Let

$$\begin{aligned} T_3^+ &= \{123, 125, 134, 145, 246, 246, 356, 356\}, \\ T_3^- &= \{124, 124, 135, 135, 236, 256, 346, 456\}. \end{aligned}$$

Then $T_3 = (X; T_3^+, T_3^-)$ is a $\mathrm{T}(2, 3, 6)$ trade which is not simple.

In this paper we are only interested in simple trades. Hence, hereafter we assume that all trades are simple.

We need to define the union operation of simple trades. Let $T_i = (X_i; T_i^+, T_i^-)$ be $\mathrm{T}(t, k, v_i)$ trades for $i = 1, 2$ with disjoint block sets, i.e. $(T_1^+ \cup T_1^-) \cap (T_2^+ \cup T_2^-) = \varnothing$. Then, the *union* of T_1 and T_2, denoted by $T_1 + T_2$ is a $\mathrm{T}(t, k, v_1 + v_2)$ trade defined as $(X_1 \cup X_2; T_1^+ \cup T_2^+, T_1^- \cup T_2^-)$.

Example 3.6 The union of disjoint trades T_1 and T_2 of Examples 3.1 and 3.2, respectively, gives a $\mathrm{T}(2, 3, 6)$ trade $T_4 = (X; T_4^+, T_4^-)$, where

$$\begin{aligned} T_4^+ &= \{123, 124, 135, 146, 156, 236, 245, 256, 345, 346\}, \\ T_4^- &= \{125, 126, 134, 136, 145, 234, 235, 246, 356, 456\}. \end{aligned}$$

Since the volume of T_4 is equal to $\binom{6}{3}/2 = 10$, T_4 at the same time represents an $\mathrm{LS}[2](2, 3, 6)$ and T_4^+ and T_4^- are the block sets of 2-(6,3,2) designs.

Let $T = (X; T^+, T^-)$ be a simple $\mathrm{T}(t, k, v)$ trade such that $T^+ \cup T^- = P_k(X)$. Since any t-subset Y of X is contained in $\binom{v-t}{k-t}$ blocks of $P_k(X)$, it follows that Y is contained in $\binom{v-t}{k-t}/2$ blocks of T^+ (T^-). Hence, (X, T^+) and (X, T^-) are t-$(v, k, \binom{v-t}{k-t}/2)$ designs and we also have an $\mathrm{LS}[2](t, k, v)$. The converse also holds, of course. Note that the volume of T is $\binom{v}{k}/2$. A simple t-$(v, k, \binom{v-t}{k-t}/2)$ design is called a *halving* design. In our approach to the existence problem of halving designs it is more natural to consider halving t-(v, k, λ) designs as $\mathrm{LS}[2](t, k, v)$ or trades of volume $\binom{v}{k}/2$.

4 Product of trades

In this section we present a description of the operation of product of trades using a number of examples. The following definition is a special case of a general notion which was introduced in [5]. The general case is considered later in Section 8. Let X_1 and X_2 be two disjoint sets of cardinality v_1 and v_2, respectively. For $\mathcal{B}_1 \subseteq P_{k_1}(X_1)$ and $\mathcal{B}_2 \subseteq P_{k_2}(X_2)$, we let

$$\mathcal{B}_1 * \mathcal{B}_2 = \{B_1 \cup B_2 | B_1 \in \mathcal{B}_1, \quad B_2 \in \mathcal{B}_2\}.$$

Let $T_i = (X_i; T_i^+, T_i^-)$ be a $\mathrm{T}(t_i, k_i, v_i)$ trade of volume s_i on X_i for $i = 1, 2$. Then, the *product* of T_1 and T_2, denoted by $T_1 * T_2$, is defined as $(X_1 \cup X_2; S^+, S^-)$, where

$$S^+ = (T_1^+ * T_2^+) \cup (T_1^- * T_2^-),$$
$$S^- = (T_1^+ * T_2^-) \cup (T_1^- * T_2^+).$$

The following is an important extension theorem concerning the product of trades. The theorem states that from trades of strength t_1 and t_2, one can produce a trade of strength $t_1 + t_2 + 1$. Since, here $+1$ is quite unexpected, we present the proof to clarify it.

Theorem 4.1 **[5]** *$T_1 * T_2$ is a $\mathrm{T}(t_1 + t_2 + 1, k_1 + k_2, v_1 + v_2)$ trade of volume $2s_1s_2$.*

Proof It is easy to see that S^+ and S^- are disjoint and $|S^+| = |S^-| = 2s_1s_2$. Therefore, we only need to show that S^+ and S^- are $(t_1 + t_2 + 1)$-balanced. Let B be a $(t_1 + t_2 + 1)$-subset of $X_1 \cup X_2$ and for $1 \le i \le 2$, define $B_i = B \cap X_i$ and $r_i = |B_i|$. With no loss of generality, we may assume that $r_1 \le t_1$ (since if $r_i > t_i$ for $i = 1, 2$, then $|B| \ge r_1 + r_2 + 2$, a contradiction).

Let x be the number of occurrences of B_1 in T_1^+ (T_1^-). Let y and z be the number of occurrences of B_2 in T_2^+ and T_2^-, respectively. Then clearly, $x(y+z)$ is the number of occurrences of B in S^+ and at the same time the number of occurrences of B in S^-. Hence, the assertion holds. □

We also need the following definition. Let $\mathcal{B} \subseteq P_{k_1}(X_1)$ be of cardinality b and let $T = (X_2; T^+, T^-)$ be a $\mathrm{T}(t, k_2, v_2)$ trade of volume s. Then, the *product* of $\mathcal{B}$ and T, denoted by $\mathcal{B} * T$, is defined as $(X_1 \cup X_2; S^+, S^-)$, where

$$S^+ = \mathcal{B} * T^+,$$
$$S^- = \mathcal{B} * T^-.$$

Theorem 4.2 **[5]** *$\mathcal{B} * T$ is a $\mathrm{T}(t, k_1 + k_2, v_1 + v_2)$ trade of volume bs.*

One may use Theorem 4.1 to construct trades of any strength t starting with trades of strength 0. Theorem 4.1 is the basis of our approach in constructing t-designs and large sets. We illustrate the product of trades and the theorems above by some examples.

Example 4.3 Let T_1 be as given in Example 3.1. Let $\mathcal{B} = \{0\}$. Then $\mathcal{B} * T_1 = (\{0, 1, \ldots, 6\}; S^+, S^-)$ is a $\mathrm{T}(2, 4, 7)$ trade by Theorem 4.2, where

$$S^+ = \{0135, 0146, 0236, 0245\},$$
$$S^- = \{0136, 0145, 0235, 0246\}.$$

Example 4.4 Let T_1 be as given in Example 3.1. Let $T_2 = (\{x, y, z, t\}; T_2^+, T_2^-)$ be a $T(1, 2, 4)$ trade where

$$T_2^+ = \{xy, zt\},$$
$$T_2^- = \{xz, yt\}.$$

Then $T_1 * T_2 = (\{1,2,\ldots,6,x,y,z,t\}; S^+, S^-)$ is a $\mathrm{T}(4,5,10)$ trade of volume 16, where

$$\begin{aligned} S^+ = \{&135xy, 146xy, 236xy, 245xy, 135zt, 146zt, 236zt, 245zt,\\ &136xz, 145xz, 235xz, 246xz, 136yt, 145yt, 235yt, 246yt\},\\ S^- = \{&135xz, 146xz, 236xz, 245xz, 135yt, 146yt, 236yt, 245yt\\ &136xy, 145xy, 235xy, 246xy, 136zt, 145zt, 235zt, 246zt\}. \end{aligned}$$

In the example above, starting with trades of strength 1 and 2, we find a trade of strength 4. Starting with trades of strength 0 and repeating this approach we can find a trade of any arbitrary strength t. This is shown in the next example.

Example 4.5 Minimal trades have a unique structure. They can be constructed through the product of trades. A minimal $T(t,k,k+t+1)$ trade is in fact the product of $t+1$ trades of strength 0. To be more specific, let

$$X = \{x_1, x_2, \ldots, x_{t+1}, y_1, y_2, \ldots, y_{t+1}, z_1, z_2, \ldots, z_{k-t-1}\}$$

be a $(k+t+1)$-set. Let $T_i = (\{x_i, y_i\}; \{x_i\}, \{y_i\})$ be a $T(0,1,2)$ trade of volume 1 for $i = 1,2,\ldots,t+1$. Then by Theorem 4.1, the product T of T_i $(1 \le i \le t+1)$ is a $T(t,t+1,2t+2)$ trade. Let $\mathcal{B} = \{z_1, z_2, \ldots, z_{k-t-1}\}$. Then by Theorem 4.2, $\mathcal{B} * T$ is a $T(t,k,k+t+1)$ trade which is minimal. For example trade T_1 of Example 3.1, is the product of trades $(\{1,2\};\{1\},\{2\})$, $(\{3,4\};\{3\},\{4\})$ and $(\{5,6\};\{5\},\{6\})$.

Trade T_2 of Example 3.2 cannot be obtained through the product operation, since then by Theorem 4.1, it should be the product of a trade of volume 1 and a trade of volume 3. But by Theorem 3.4, the trade of volume 1 has strength 0 and it is of the form $(\{x,y\};\{x\},\{y\})$ where $x, y \in \{1,2,\ldots,6\}$. Therefore every block of T_2 should contain x or y which is not the case.

Combining two operations of union and product of trades we can construct many trades. In [8], this method has been used to construct simple $\mathrm{T}(2,3,v)$ trades for any even foundation size v and any possible volume.

5 Recursive constructions

In this section we make use of the two operations defined in the previous sections, i.e. the product and the union operations of trades, to obtain some recursive constructions for large sets of size 2. We note that although the constructions are given for large sets of size 2, however they can easily be extended to any size N. We will discuss the general case in Section 8.

A description of the approach is as follows: We construct some block disjoint $\mathrm{T}(t,k,v)$ trades on a v-set X using the product operation and then take their union. If the resulting trade covers all k-subsets of X, then we have an $\mathrm{LS}[2](t,k,v)$. The following lemma provides a formal statement for a later use.

Lemma 5.1 *If there exist* $\mathrm{T}(t,k,v)$ *trades* $T_i = (X; T_i^+, T_i^-)$ $(1 \le i \le n)$ *such that* $T_i^+ \cup T_i^-$ *partition* $P_k(X)$*, then there exists an* $LS[2](t,k,v)$*.*

Proof Let

$$T^+ = \bigcup_{i=1}^{n} T_i^+$$
$$T^- = \bigcup_{i=1}^{n} T_i^-.$$

Then $T = (X; T^+, T^-)$ is a $\mathrm{T}(t,k,v)$ trade such that $T^+ \cup T^- = P_k(X)$. Therefore, (X, T^+) and (X, T^+) are halving designs and we have an $\mathrm{LS}[2](t,k,v)$. □

The recursive constructions we present in this section stem from some binomial identities. We start with the following simple one:

$$\binom{v}{k} = \binom{v}{v-k},$$

which follows from the trivial one-to-one correspondence between k-subsets and $(v-k)$-subsets of a v-set. Translating this to the language of large sets we obtain Theorem 2.3.

The next identity is

$$\binom{v}{k} = \binom{v-1}{k} + \binom{v-1}{k-1},$$

which is obtained by counting the number of k-subsets of a v-set in two ways. The left hand side is straightforward since the number of k-subsets of a v-set is $\binom{v}{k}$. The right hand side follows from counting first k-subsets which do not contain a fixed point x and then counting k-subsets which contain x. The identity and its proof suggests the following for large sets.

Theorem 5.2 *There exists an* $LS[2](t,k,v)$ *if and only if there exist an* $LS[2](t,k,v-1)$ *and an* $LS[2](t,k-1,v-1)$.

Proof From $\mathrm{LS}[2](t,k,v)$ we obtain an $\mathrm{LS}[2](t,k,v-1)$ and an $\mathrm{LS}[2](t,k-1,v-1)$ using Theorem 2.2.

For the converse, let X be a $(v-1)$-set and $x \notin X$. From the assumption, there is a $\mathrm{T}(t,k,v)$ trade $T_1 = (X \cup \{x\}; T_1^+, T_1^-)$ such that $T_1^+ \cup T_1^- = P_k(X)$ and also a $\mathrm{T}(t,k-1,v-1)$ trade $T_2 = (X; T_2^+, T_2^-)$ such that $T_2^+ \cup T_2^- = P_{k-1}(X)$. Let $T_3 = \{x\} * T_2$. Then by Theorem 4.2, T_3 is a $\mathrm{T}(t,k,v)$ trade. Note that T_i^+ and T_i^- $(i = 1, 3)$ partition $P_k(X \cup \{x\})$. Now from Lemma 5.1 we obtain an $\mathrm{LS}[2](t,k,v)$. □

The following theorem is a consequence of Theorem 5.2 and an induction argument.

Theorem 5.3 *If there exist* $LS[2](t,k+i,v)$ *for all* $0 \le i \le l$, *then there exist* $LS[2](t,k+i,v+j)$ *for all* $0 \le j \le i \le l$.

The following identity is more involved:

$$\binom{v+1}{a+b+1} = \sum_{i=a}^{v-b} \binom{i}{a}\binom{v-i}{b}.$$

It is obtained using the double counting suggested by the following lemma.

Lemma 5.4 *We have*

$$P_{a+b+1}(\{1,2,\ldots,v+1\}) = \bigcup_{i=a}^{v-b} (P_a(\{1,2,\ldots,i\}) * \{i+1\} * P_b(\{i+2,i+3,\ldots,v+1\}).$$

Proof Sort $(a+b+1)$-subsets of $\{1,2,\ldots,v+1\}$ in the lexicographic order and partition them by looking at the elements in the position $a+1$. $\square$

We demonstrate by an example how the lemma can be utilized to construct large sets.

Example 5.5 We construct an LS[2]$(2,3,10)$. To do this we first construct an LS[2]$(2,7,10)$ and then using Theorem 2.3 we obtain an LS[2]$(2,3,10)$. Let $a=b=3$ and $v=9$ in Lemma 5.4 and let $X=\{1,2,\ldots,10\}$. We have

$$P_7(X) = \bigcup_{i=3}^{6} \mathcal{B}_i,$$

where

$$\begin{aligned}
\mathcal{B}_3 &= P_3(\{1,2,3\}) * \{4\} * P_3(\{5,6,\ldots,10\},\\
\mathcal{B}_4 &= P_3(\{1,2,3,4\}) * \{5\} * P_3(\{6,7,\ldots,10\},\\
\mathcal{B}_5 &= P_3(\{1,2,\ldots,5\}) * \{6\} * P_3(\{7,8,9,10\},\\
\mathcal{B}_6 &= P_3(\{1,2,\ldots,6\}) * \{7\} * P_3(\{8,9,10\}.
\end{aligned}$$

We show that there are T$(2,7,10)$ trades $T_i=(X;T_i^+,T_i^-)$ for $i=3,4,5,6$ such that $T_i^+ \cup T_i^- = \mathcal{B}_i$ and hence by Lemma 5.1, there is an LS[2]$(2,7,10)$. In Example 3.6, we constructed an LS[2]$(2,3,6)$. From this and Theorem 2.2, we obtain an LS[2]$(1,3,5)$ and an LS[2]$(0,3,4)$. From LS[2]$(2,3,6)$ we find a T$(2,3,6)$ trade $T_1=(Y;T_1^+,T_1^-)$, where $Y=\{5,6,\ldots,10\}$. Now let $T_3=P_3(\{1,2,3\})*\{4\}*T_1$ which is, by Theorem 4.2, a T$(2,7,10)$ trade. T_6 is constructed in the same way. From LS[2]$(0,3,4)$ we find a T$(0,3,4)$ trade $T_1=(Y;T_1^+,T_1^-)$, where $Y=\{1,2,3,4\}$ and from LS[2]$(1,3,5)$ we find a T$(1,3,5)$ trade $T_2=(Z;T_2^+,T_2^-)$, where $Z=\{6,7,\ldots,10\}$. Now let $T_4=T_1*\{5\}*T_2$ which is, by Theorems 4.1 and 4.2, a T$(2,7,10)$ trade. T_5 is constructed in a similar way.

The partition given by Lemma 5.4, has been used to obtain a recursive construction for large sets of prime sizes in [34].

Here is another example of a useful binomial identity:

$$\binom{u+v+1}{k} = \sum_{i=0}^{k} \binom{u-i}{k-i}\binom{v+i}{i},$$

with a proof given in the next lemma.

Lemma 5.6 *Let $X = \{1, \ldots, u+v+1\}$ and let $X_j = \{1, \ldots, j\}$ and $Y_j = X \setminus X_j$ for $j = 1, \ldots, u+v+1$. Assume that*

$$\mathcal{B}_i = P_{k-i}(X_{u-i}) * P_i(Y_{u-i+1}), \qquad 0 \le i \le k.$$

Then the sets $\mathcal{B}_i$ partition $P_k(X)$.

Proof Let $0 \le j < i \le k$ and $A \in \mathcal{B}_i$. Then $|A \cap X_{u-i}| = k - i$ and

$$\begin{aligned} |A \cap X_{u-j}| &\le |A \cap X_{u-i}| + |X_{u-j} \setminus X_{u-i}| - 1 \\ &= k - j - 1. \end{aligned}$$

Therefore, $A \notin \mathcal{B}_j$. It yields that all $\mathcal{B}_i$ are mutually disjoint.

Now let $A \in P_k(X)$. Let $0 \le i \le k$ be the smallest integer such that $|A \cap X_{u-i}| \ge k - i$. Then $|A \cap X_{u-i+1}| \le k - i + 1$ and therefore,

$$\begin{aligned} k - i &\le |A \cap X_{u-i}| \\ &\le |A \cap X_{u-i+1}| \\ &\le k - i. \end{aligned}$$

Hence, $|A \cap X_{u-i}| = |A \cap X_{u-i+1}| = k - i$ and $A \in \mathcal{B}_i$. □

An extension of Lemma 5.6 is given in [2] (see also [25]). For our purpose, this simplified version is well suited. Before presenting the construction, we give an example.

Example 5.7 We construct an LS[2](2, 3, 10) from an LS[2](2, 3, 6) using the partition given in Lemma 5.6 (compare to the construction given in Example 5.5). Let $u = 6, v = 4$ and $k = 3$ in Lemma 5.6 and let $X = \{1, 2, \ldots, 10\}$. We have

$$P_3(X) = \bigcup_{i=0}^{3} \mathcal{B}_i,$$

where

$$\begin{aligned} \mathcal{B}_0 &= P_3(\{1, 2, \ldots, 6\}), \\ \mathcal{B}_1 &= P_2(\{1, 2, \ldots, 5\}) * P_1(\{7, 8, 9, 10\}), \\ \mathcal{B}_2 &= P_1(\{1, 2, 3, 4\}) * P_2(\{6, 7, \ldots, 10\}), \\ \mathcal{B}_3 &= P_3(\{5, , 6, \ldots, 10\}). \end{aligned}$$

We show that there are T(2, 3, 10) trades $T_i = (X; T_i^+, T_i^-)$ for $i = 0, 1, 2, 3$ such that $T_i^+ \cup T_i^- = \mathcal{B}_i$ and hence by Lemma 5.1, there is an LS[2](2, 3, 10). In Example 3.6, we constructed an LS[2](2, 3, 6). From this and Theorem 2.2, we obtain an LS[2](1, 2, 5) and an LS[2](0, 1, 4). From LS[2](2, 3, 6) we find a T(2, 3, 6) trade $T_0 = (Y; T_0^+, T_0^-)$, where $Y = \{1, 2, \ldots, 6\}$. Similarly, we construct a T(2, 3, 6) trade $T_3 = (Z; T_3^+, T_3^-)$, where $Z = \{5, 6, \ldots, 10\}$. From LS[2](1, 2, 5) we have a T(1, 2, 5) trade $T_4 = (\{1, 2, \ldots, 5\}; T_4^+, T_4^-)$ and from LS[2](0, 1, 4) we find a T(0, 1, 4) trade $T_5 = (\{7, 8, 9, 10\}; T_5^+, T_5^-)$. Now let $T_1 = T_4 * T_5$ which by Theorem 4.1, is a T(2, 3, 10) trade. T_2 is constructed in the same way.

Now using the partition provided by Lemma 5.6, we present an important recursive construction.

Theorem 5.8 [5] *If $LS[2](t, i, v+i)$ exist for all $t+1 \leq i \leq k$ and an $LS[2](t, k, u)$ also exists, then $LS[2](t, k, u+l(v+1))$ exist for all $l \geq 1$.*

Proof It suffices to prove the theorem for $l = 1$. Then the general case easily follows by an induction on l.

Consider the partition given in Lemma 5.6. We show that there exist $\mathrm{T}(t, k, u+v+1)$ trades $T_i = (X; T_i^+, T_i^-)$ for $1 \leq i \leq k$ such that $T_i^+ \cup T_i^- = \mathcal{B}_i$ and hence by Lemma 5.1, there is an $\mathrm{LS}[2](t, k, u+v+1)$.

Let $0 \leq i \leq k$. Recall that

$$\mathcal{B}_i = P_{k-i}(X_{u-i}) * P_i(Y_{u-i+1}).$$

First assume that $0 \leq i \leq t$. By $\mathrm{LS}[2](t, k, u)$ which exists by the assumption and Theorem 2.2, we acquire $\mathrm{LS}[2](t-i, k-i, u-i)$ and from this we obtain a $\mathrm{T}(t-i, k-i, u-i)$ trade $T = (X_i; T^+, T^-)$ such that $T^+ \cup T^- = P_{k-i}(X_{u-i})$. By $\mathrm{LS}[2](t, t+1, v+t+1)$ which comes from the assumption and Theorem 2.2, we also obtain an $\mathrm{LS}[2](i-1, i, v+i)$. Since $|Y_{u-i+1}| = v+i$, there is a $\mathrm{T}(i-1, i, v+i)$ trade $T' = (Y_{u-i+1}; T'^+, T'^-)$ such that $T'^+ \cup T'^- = P_i(Y_{u-i+1})$. Now let $T_i = T * T'$ which is a $\mathrm{T}(t, k, u+v+1)$ trade by Theorem 4.1. Next let $i > t$. By the assumption we have an $\mathrm{LS}[2](t, i, v+i)$ and so there is a $\mathrm{T}(t, i, v+i)$ trade $T = (Y_{u-i+1}; T^+, T^-)$ such that $T^+ \cup T^- = P_i(Y_{u-i+1})$. Now let $T_i = P_{k-i}(X_{u-i}) * T$ which is $\mathrm{T}(t, k, u+v+1)$ trade by Theorem 4.2. □

More recursive constructions have been found using this approach. The reader is referred to [2, 3, 25, 27, 34]. In Section 8, we return to these constructions when we mention large sets of any size.

6 Halving designs

In Section 3, we observed that halving t-(v, k, λ) designs, large sets $\mathrm{LS}[2](t, k, v)$ and $\mathrm{T}(t, k, v)$ trades of volume $\binom{v}{k}/2$ are in principle the same objects. In the sequel, we consider $\mathrm{LS}[2](t, k, v)$ as halving designs.

From (2.1), we recall that the parameters of an $\mathrm{LS}[2](t, k, v)$ satisfy the following necessary conditions:

$$2 \Big| \binom{v-i}{k-i}, \qquad 0 \leq i \leq t. \tag{6.1}$$

A triple (t, k, v) satisfying (6.1) is called *feasible*. From Theorem 2.4, we have the following lemma.

Lemma 6.1 *Let $0 \leq t \leq k \leq v$. Then (t, k, v) is feasible if and only if there is a positive integer z such that $t \leq (v)_{2^z} < (k)_{2^z}$.*

The above characterization can be rephrased as follows.

Lemma 6.2 *Let $0 \leq t \leq k \leq v$. Then (t, k, v) is feasible if and only if $k = m2^z + r$ and $v = n2^z + s$ for some integers r, s, z, m, n such that $t \leq s < r < 2^z$.*

A long-standing conjecture of Hartman [16] known as the halving conjecture states that the necessary conditions (6.1) are sufficient for the existence of an $\mathrm{LS}[2](t,k,v)$. Formally we have the following.

Halving conjecture Let $0 \le t \le k \le v$. Then there exists an $\mathrm{LS}[2](t,k,v)$ if and only if

$$2 \Big| \binom{v-i}{k-i}, \qquad 0 \le i \le t.$$

Along this line, the first author (GBK) proposed its analogue for large sets of size 3 [4].

Conjecture Let $0 \le t \le k \le v$. Then there exists an $\mathrm{LS}[3](t,k,v)$ if and only if

$$3 \Big| \binom{v-i}{k-i}, \qquad 0 \le i \le t.$$

For some results concerning this conjecture, see [26, 34].

In spite of some results concerning the existence of halving designs, the conjecture is still wide open and seems to be far from being resolved in the near future. The following theorem which follows from the recursive constructions given in Section 5 and an induction argument provides a strategy to tackle halving conjecture. The large sets needed in the following theorem are called *root cases* [27].

Theorem 6.3 **[1]** *Let t, k and s be positive integers such that $2^s - 1 \le t < 2^{s+1} - 1$ and $t < k$. Suppose that for every j and n such that $0 \le j \le \lceil t/2 \rceil$ and $t + 1 \le 2^n + j \le k$, there exists an $LS[2](t, 2^n + j, 2^{n+1} + t)$. Then for any integer $v > k$ such that the triple (t,k,v) is feasible, there exists an* $\mathrm{LS}[2](t,k,v)$.

The most outstanding result on halving designs is due to Ajoodani-Namini which establishes the halving conjecture for $t = 2$ in [1]. By the theorem above in order to prove the conjecture for $t = 2$, it suffices to find two infinite families $\mathrm{LS}[2](2, 2^n + 1, 2^{n+1} + 2)$ and $\mathrm{LS}[2](2, 2^n, 2^{n+1} + 2)$. In fact both families exist. The existence of the first family is a result of two well known theorems. By Baranyai's theorem [7], there exists an $\mathrm{LS}[2](1, 2^n, 2^{n+1} + 1)$ and by Alltop's theorem [6] it can be extended to $\mathrm{LS}[2](2, 2^n + 1, 2^{n+1} + 2)$. Ajoodani-Namini constructed the second family using the approach of products of trades [1]. His construction is rather complicated and leaves little hope for an extension to higher values of t. For $t > 2$, there are some partial results. For $t = 3$, the conjecture has been settled for infinitely many values of k [2, 25]. For some other results on small halving designs, see [30].

Now we present a general view or a road map on how to attack halving conjecture: One way is to tackle the problem for any given t. In this scenario, for the settlement of the conjecture for given t, by Theorem 6.3, one has to construct the root cases $\mathrm{LS}[2](t, 2^n + j, 2^{n+1} + t)$, where $0 \le j \le t/2$. The other possible way, for our ambitious champion, is to construct another class of root cases, namely $\mathrm{LS}[2](2^n - 2, 2^n - 1, 2^{n+1} - 2)$ which resolves halving conjecture for all t [26]. We note that from this class, large sets are known only for $n = 1, 2, 3$ [13, 24, 29, 31].

7 N-Legged trades

In this section we discuss a generalization of trades recently called N-legged trades. We note that (N,t)-partitionable sets which will be presented in the next section are in fact simple N-legged trades and have been utilized in recursive constructions of large sets in the past.

Let v,k and t be integers such that $v \geq k \geq t \geq 0$ and let X be a v-set. A $T[N](t,k,v)$ *trade* (briefly *N-legged trade*), is a pair $T=(X,\{T_1,T_2,\ldots,T_N\})$ such that for $i \neq j$, $(X,\{T_i,T_j\})$ is a $\mathrm{T}(t,k,v)$ trade. For the sake of simplicity, we write $T=(X;T_1,T_2,\ldots,T_N)$. Note that any $\mathrm{LS}[N](t,k,v)$ is in fact a $\mathrm{T}[N](t,k,v)$ trade, however the converse is not true. We present some examples. The first two ones are taken from [18].

Example 7.1 Let $X=\{1,2,\ldots,7\}$. Let

$$\begin{aligned}
T_1 &= \{123,167,247,256,346,357\},\\
T_2 &= \{127,136,235,246,347,567\},\\
T_3 &= \{126,137,234,257,356,467\}.
\end{aligned}$$

Then $(X;T_1,T_2,T_3)$ is a $\mathrm{T}[3](2,3,7)$ trade.

Example 7.2 Let $X=\{1,2,\ldots,8\}$. Let

$$\begin{aligned}
T_1 &= \{123,145,167,248,257,346,378,568\},\\
T_2 &= \{124,136,157,237,258,348,456,678\},\\
T_3 &= \{125,137,146,234,278,368,458,567\},\\
T_4 &= \{127,134,156,238,245,367,468,578\}.
\end{aligned}$$

Then $(X;T_1,T_2,T_3,T_4)$ is a $\mathrm{T}[4](2,3,8)$ trade.

Example 7.3 Let $X=\{1,2,\ldots,9\}$. Let

$$\begin{aligned}
T_1 &= \{124,138,157,169,237,259,268,349,356,458,467,789\},\\
T_2 &= \{129,136,145,178,235,248,267,347,389,469,568,579\},\\
T_3 &= \{123,148,159,167,249,256,278,346,358,379,457,689\},\\
T_4 &= \{126,135,147,189,234,258,279,369,378,459,468,567\},\\
T_5 &= \{128,137,149,156,239,246,257,345,368,478,589,679\},\\
T_6 &= \{125,134,168,179,238,247,269,359,367,456,489,578\},\\
T_7 &= \{127,139,146,158,236,245,289,348,357,479,569,678\}.
\end{aligned}$$

Then $(X;T_1,T_2,\ldots,T_7)$ is a $\mathrm{T}[7](2,3,9)$ trade. Note that (X,T_i) is a 2-(9,3,1) design and hence we also have an $\mathrm{LS}[9](2,3,9)$.

As an analogue of Lemma 3.3, we have the following lemma for N-legged trades.

Lemma 7.4 *A* $\mathrm{T}[N](t,k,v)$ *trade* T *is also a* $\mathrm{T}[N](t',k,v)$ *trade for any* $0 \leq t' < t$.

Let $T = (X; T_1, T_2, \ldots, T_N)$ be a T$[N](t,k,v)$ trade. If we take $t' = 0$ in this lemma, then we find that the number of blocks in T_i for $1 \le i \le N$ is fixed and is called the *volume* of T. By taking $t' = 1$, it turns out that the set of points covered by any T_i is the same. This set is called the *foundation* of T. Trades with distinct blocks are said to be *simple*.

Not much is known about N-legged trades. In [14] some computational results for N-legged trades with $t = 2$ and $k = 3$ were obtained. In [12], 3-legged trades have been studied. Here, we briefly review the results of [12]. First no analogue of Theorem 3.4 for N-legged trades is known. It is shown that the minimum foundation size of a simple T$[3](2,3,v)$ trade is 7. In fact there exists exactly one simple T$[3](2,3,7)$ trade which has volume 6 and is given in Example 7.1. There are exactly 7 nonisomorphic simple T$[3](2,3,8)$ trades with foundation size 8. Three of these are of volume 8, one of volume 10 and five of volume 12. Also the maximum possible volume for simple T$[3](2,3,v)$ trades are found for $v \equiv 1,3,4 \pmod 6$ and $v \equiv 2 \pmod 9$.

There are many questions concerning N-legged trades. The main question is about the minimum volume and minimum foundation size of a T$[N](t,k,v)$.

It is possible to extend the definition of product of trades to N-legged trades. However, we think that it is more natural to consider the product operation in the context of (N,t)-partitionable sets. We discuss this matter in the next section.

8 (N,t)-Partitionable sets

A powerful approach for the construction of large sets is obtained through the notion of (N,t)-partitionable sets which was first introduced in [5]. The notion of (N,t)-partitionable sets is a generalization the notion of trades and indeed they are equivalent to simple N-legged trades discussed in the previous section.

Let v, k and t be integers such that $v \ge k \ge t \ge 0$ and let X be a v-set. Let $\mathcal{B}_1, \mathcal{B}_2 \subseteq P_k(X)$. We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are *t-equivalent* if every t-subset of X appears in the same number of blocks of $\mathcal{B}_1$ and $\mathcal{B}_2$. If there exists a partition of $\mathcal{B} \subseteq P_k(X)$ into N mutually t-equivalent subsets, then $\mathcal{B}$ is called an *(N,t)-partitionable set*. It is easily seen that $\mathcal{B}$ can be used to obtain a T$[N](t,k,v)$ trade. The legs of the trade will be the parts of partition of $\mathcal{B}$. Here is an example.

Example 8.1 This example is from [12]. Let

$$\begin{aligned}\mathcal{B} = \{&123, 124, 125, 136, 137, 145, 146, 157, 167, 234, 237, 248,\\ &257, 258, 278, 346, 348, 368, 378, 456, 458, 567, 568, 678\}.\end{aligned}$$

Consider the following partition of $\mathcal{B}$:

$$\begin{aligned}T_1 &= \{123, 145, 167, 248, 257, 346, 378, 568\},\\ T_2 &= \{124, 136, 157, 237, 258, 348, 456, 678\},\\ T_3 &= \{125, 137, 146, 234, 278, 368, 458, 568\}.\end{aligned}$$

T_1, T_2 and T_3 are mutually 2-equivalent. Therefore, $\mathcal{B}$ is (3,2)-partitionable set. Note that $(\{1,2,\ldots,8\}; T_1, T_2, T_3)$ is a T$[3](2,3,8)$ trade.

Now we present two important lemmas concerning (N,t)-partitionable sets. The first is a trivial one while the other is unexpected. Let X_1 and X_2 be two disjoint sets and let $\mathcal{B}_i \subseteq P_{k_i}(X_i)$ for $i = 1, 2$. Then from Section 4 recall that

$$\mathcal{B}_1 * \mathcal{B}_2 = \{B_1 \cup B_2 |\ B_1 \in \mathcal{B}_1,\ B_2 \in \mathcal{B}_2\}.$$

Lemma 8.2 **[5]** (i) *t-equivalence implies i-equivalence for all $0 \le i \le t$.*
(ii) *The union of disjoint (N,t)-partitionable sets is again an (N,t)-partitionable set.*

Lemma 8.3 **[5]** *Let X_1 and X_2 be two disjoint sets and let $\mathcal{B}_i \subseteq P_{k_i}(X_i)$ for $i = 1, 2$. Suppose that $\mathcal{B}_1$ is (N,t_1)-partitionable. Then*

(i) *$\mathcal{B}_1 * \mathcal{B}_2$ is (N,t_1)-partitionable.*
(ii) *If $\mathcal{B}_2$ is also (N,t_2)-partitionable, then $\mathcal{B}_1 * \mathcal{B}_2$ is (N,t_1+t_2+1)-partitionable.*

We now explain the construction which is used in Lemma 8.3 (ii). Let $T_1, T_2, \ldots, T_N$ be a partition of $\mathcal{B}_1$ into N mutually t_1-equivalent subsets and let $S_1, S_2, \ldots, S_N$ be a partition of $\mathcal{B}_2$ into N mutually t_2-equivalent subsets. We need to find a partition $R_1, R_2, \ldots, R_N$ of $\mathcal{B}_1 * \mathcal{B}_2$ into N mutually (t_1+t_2+1)-equivalent subsets. Consider the partition $T_i * S_j$, $1 \le i,j \le N$ of $\mathcal{B}_1 * \mathcal{B}_2$. Let L be a Latin square of order n with entries from $\{1, 2, \ldots, N\}$. Define

$$R_f = \bigcup_{L_{ij}=f} T_i * S_j,$$

for $1 \le f \le N$. We give an example taken from [18] to clarify the construction.

Example 8.4 Let

$$\begin{aligned}\mathcal{B}_1 &= \{1, 2, 3\},\\ \mathcal{B}_2 &= \{45, 46, 47, 56, 57, 67\}.\end{aligned}$$

$\mathcal{B}_1$ is (3,0)-partitionable with the partition $T_1 = \{1\}$, $T_2 = \{2\}$ and $T_3 = \{3\}$. $\mathcal{B}_2$ is (3,1)-partitionable with the partition $S_1 = \{45, 67\}$, $S_2 = \{46, 57\}$ and $S_3 = \{47, 56\}$. Consider the following Latin square:

$$\begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{matrix}$$

Then we have

$$\begin{aligned}R_1 &= (T_1 * S_1) \cup (T_2 * S_2) \cup (T_3 * S_3) = \{145, 167, 246, 257, 347, 356\},\\ R_2 &= (T_1 * S_2) \cup (T_2 * S_3) \cup (T_3 * S_1) = \{146, 157, 247, 256, 345, 367\},\\ R_3 &= (T_1 * S_3) \cup (T_2 * S_1) \cup (T_3 * S_2) = \{147, 156, 245, 267, 346, 357\}.\end{aligned}$$

R_1, R_2 and R_3 provide a partition of $\mathcal{B}_1 * \mathcal{B}_2$ into 3 mutually 2-equivalent subsets. Note that $(\{1, 2, \ldots, 7\}; R_1, R_2, R_3)$ is isomorphic to the unique T[3]$(2, 3, 7)$ trade given in Example 7.1.

The approach of (N,t)-partitionable sets for constructing large sets is based on Lemmas 8.2 and 8.3. Suppose that we are looking for an LS$[N](t,k,v)$ on a v-set X. We try to partition $P_k(X)$ in such a way that each part of the partition is an (N,t)-partitionable set. If this done, then by Lemma 8.2, $P_k(X)$ will be an (N,t)-partitionable set which means that we have obtained an LS$[N](t,k,v)$. We have previously used the approach for large sets of size 2 in Section 5. Here, we give an example of large sets of size 3 taken from [26].

Example 8.5 We construct an LS$[3](2,12,29)$ from LS$[3](2,3,11)$, LS$[3](2,7,15)$, LS$[3](2,8,16)$ and LS$[3](2,12,20)$. Note that there is known no other construction method for LS$[3](2,12,29)$. Let $X=\{1,2,\ldots,29\}$ and let $u=20, v=8$ and $k=12$ in Lemma 5.6. Then we have

$$P_{12}(X)=\bigcup_{i=0}^{12}\mathcal{B}_i,$$

where

$$\mathcal{B}_i=P_{12-i}(\{1,\ldots,20-i\})*P_i(\{22-i,\ldots,29\}),\quad 0\le i\le 12.$$

$\mathcal{B}_0$ and $\mathcal{B}_{12}$ are (3,2)-partitionable sets since there exists an LS$[3](2,12,20)$. By Theorem 2.2, there exist LS$[3](1,2,10)$, LS$[3](0,1,9)$, LS$[3](1,6,14)$, LS$[3](0,10,18)$ and LS$[3](1,11,19)$. Using Lemma 8.3, it is an easy task to see that all the remaining $\mathcal{B}_i$ are also (3,2)-partitionable sets. For example, consider $\mathcal{B}_2$ and $\mathcal{B}_3$. $P_{10}(\{1,\ldots,18\})$ is (3,0)-partitionable since there is an LS$[3](0,10,18)$. $P_2(\{20,\ldots,29\})$ is (3,1)-partitionable since LS$[3](1,2,10)$ exists. Now by Lemma 8.3, $\mathcal{B}_2$ is (3,2)-partitionable. By the existence of an LS$[3](2,3,11)$, $P_3(\{19,\ldots,29\})$ is a (3,2)-partitionable set and so is $\mathcal{B}_3$ by Lemma 8.3. The remaining $\mathcal{B}_i$ are dealt with in similar ways. Hence, by Lemma 8.2, $P_{12}(X)$ is (3,2)-partitionable and so an LS$[3](2,12,29)$ is constructed.

The recursive constructions for large sets of size 2 given in Section 5 are easily extended to large sets of any size. More constructions can be found in [2, 3, 27, 25, 34]. Here we present two important constructions by Ajoodani-Namini [3] for large sets of prime size p.

Theorem 8.6 [3] *If there exists an LS$[p](t,k,v-1)$, then there exist LS$[p](t+1,pk+i,pv+j)$ for all $0\le j<i\le p-1$.*

Theorem 8.7 [3, 35] *If there exists an LS$[p](t,k,v-1)$, then there exist LS$[p](t,pk+i,pv+j)$ exist for all $-p\le j<i\le p-1$.*

The above theorems could be utilized to produce a large number of infinite families of large sets. As Ajoodani-Namini [2] has noted, these theorems are unique in design theory in the sense that they impose no further conditions on the parameters. By this, we mean that given any large set (whatever the parameters are), using these theorems one can construct infinite families of large sets. The reason for it is that any large set of size N leads to a large set of size p for any prime divisor p of N. Theorem 8.6 is specially interesting since it proves Teirlinck's theorem [36] on the existence of simple t-designs for all t. It also has the extra merit that one can produce t-designs on point sets which are very small compared to those of Teirlinck's.

9 A linear algebraic approach to trades and designs

In this section we present a linear algebraic approach to the study of trades and designs. By giving two applications we show the usefulness of this representation of trades. Let $0 \leq t \leq k \leq v-t$ and and let X be a v-set. Order $P_t(X)$ and $P_k(X)$ lexicographically (or with any other ordering). Let $W_{tk}(v)$ be a $\binom{v}{t} \times \binom{v}{k}$ $(0,1)$-matrix whose rows and columns are indexed by the elements of $P_t(X)$ and $P_k(X)$, respectively, and for a t-subset T and a k-subset K, $W_{tk}(v)(T,K) = 1$ if and only if $T \subseteq K$. The matrix $W_{tk}(v)$ is an *inclusion matrix* which is often called the *Wilson matrix* since it was first introduced and used by Wilson [37]. We simply write W_{tk} instead of $W_{tk}(v)$ if there is no risk of confusion.

Given a collection of elements of $P_k(X)$, a $\binom{v}{k}$ column vector F can be associated with it: $F = (f_1, f_2, \ldots, f_{\binom{v}{k}})$ where f_i is the frequency of ith element of $P_k(X)$ in this collection. Conversely, for a given column vector F of size $\binom{v}{k}$ with nonnegative integers, a collection of the elements of $P_k(X)$ can be associated with it by taking f_i copies of the ith element of $P_k(X)$. Also for a given $S = \{S_1, S_2\}$, where S_1 and S_2 are two disjoint collections of elements each from $P_k(X)$, an integral $\binom{v}{k}$ column vector F can be associated with it whose ith entry is f_i if the ith block of $P_k(X)$ appears f_i times in S_1 or $-f_i$ if the ith block of $P_k(X)$ appears f_i times in S_2.

A $\binom{v}{k}$ column vector F with nonnegative integers represents a t-design if $W_{tk}F = \lambda J$, where λ is a positive integer and J is the all one vector. If we let negative entries in F, then we have a *signed t-design.*

A $\binom{v}{k}$ integral column vector F is a $\mathrm{T}(t,k,v)$ trade if and only if $W_{tk}F = 0$. The positive components of F identify the frequencies of the blocks in T^+ (T^-) and the negative components (sign ignored) identify the frequencies of the blocks in T^- (T^+). If F_1 and F_2 represent two $\mathrm{T}(t,k,v)$ trades based on X, then $F_1 + F_2$ as well as nF_1 ($n \in \mathbb{Z}$) are also trades. Therefore, the set of all $\mathrm{T}(t,k,v)$ trades forms a free $\mathbb{Z}$-module. It is well known that the rank of W_{tk} is $\binom{v}{t}$ and the null space of W_{tk} that generates all $\mathrm{T}(t,k,v)$ trades has dimension $\binom{v}{k} - \binom{v}{t}$ [15, 37].

If F_1 and F_2 represent two t-(v,k,λ) designs based on X, then $F_1 - F_2$ is a trade. Therefore, in principle all t-(v,k,λ) designs can be generated by using trades and any given signed t-(v,k,λ) design. To do this we need to find first a signed t-(v,k,λ) design F (which is easy to find) and then combining it with all trades G such that $F+G$ is a nonnegative vector. This shows the importance of trades in the study of t-designs and suggests that to study t-designs one may investigate trades.

Different bases have been presented in the literature for the $\mathbb{Z}$-module of trades. A survey is given in [23]. In [20] a triangular basis for trades is given. All trades in this basis are minimal. This paper also gives an algorithm based on this basis to find halving designs in triple systems.

An interesting basis for trades is the so called the *standard basis* given in [23] which follows from the basis given in [20]. The $\binom{v}{k} - \binom{v}{t}$ trades of the standard basis constitute the columns of a matrix $M^v_{t,k}$ which has the following block structure:

$$M^v_{t,k} = \left[\begin{array}{c} I \\ \hline M^v_{t,k} \end{array} \right].$$

The rows corresponding to I are indexed by the so-called *starting blocks* and the remaining rows by the *non-starting blocks* [20]. This basis has many interesting properties yet to be explored. The standard basis for $\mathrm{T}(2,3,6)$ trades is given in Table 1. In this table the first column shows the starting and non starting blocks, the next five columns show the five trades in the basis and the last column is a signed 2-(6,3,2) design obtained in a way described in the following.

Table 1: The standard basis for $\mathrm{T}(2,3,6)$ trades

123	1					0
124		1				0
125			1			0
134				1		0
135					1	0
126	−1	−1	−1			2
136	−1			−1	−1	2
145	−1	−1	−1	−1	−1	3
146	1		1		1	−1
156	1	1		1		−1
234	−1	−1		−1		2
235	−1		−1		−1	2
236	1	1	1	1	1	−2
245	1			1	1	−1
345	1	1	1			−1
246					−1	1
256				−1		1
346			−1			1
356		−1				1
456	−1					1

We present two applications of the standard basis. Note that the feasibility conditions for the existence of a t-(v,k,λ) design given in Section 2 are sufficient for the existence of a signed t-(v,k,λ) design [15, 28, 37]. Signed designs are useful in the study of t-designs. To construct designs using the so called trade off method, one starts with a signed design and then tries to eliminate negative entries by adding suitable trades. We show how the standard basis is used to produce a signed design. To find a signed t-(v,k,λ) design, it is enough to sum up all the columns of $M^v_{t,k}$, then subtract it from the vector J and finally divide the resulting vector by a suitable coefficient [28]. Note that all entries in this signed design corresponding to starting blocks are zero. This signed design can sometimes be converted to a t-design by adding a suitable trade. As an example a signed 2-(6,3,2) design obtained by this method is shown in the last column of Table 1.

A halving design is equivalent to a trade F whose entries are ± 1. Therefore, to find a halving design one can use the standard basis and take a combination of columns with coefficients 1 or -1 and then check whether the resulting trade is simple and has no zero entry. This approach is effective since the standard basis has the recursive structure

$$M^v_{t,k} = \left[\begin{array}{cc} I & 0 \\ 0 & I \\ \cline{1-1} M^{v-1}_{t-1,k-1} & 0 \\ \cline{2-2} N & M^{v-1}_{t,k} \end{array} \right],$$

which suggests that to find a halving t-(v, k, λ) design through extending halving $(t-1)$-$(v-1, k-1, \lambda)$ designs. A detailed description of this method can be found in [13] where it has been successfully used to obtain new 6-(14,7,4) designs.

10 Concluding remarks

In this paper we have presented a rather pedagogical review of the application of trades in constructing halving t-designs. We have also considered the notion of (N, t)-partitionable sets as a generalization of trades and have shown how some powerful recursive constructions can be obtained for large sets of t-designs. We hope that we have been able to draw the attention of the reader to the power of the approach of (N, t)-partitionable sets. There are some open problems to undertake further research in the future. The first problem is to find other binomial identities besides the ones given in Section 5 which correspond to recursive constructions for large sets. Halving conjecture for $t = 3$ is another problem which seems to be hard. One may think that the similar problem for large sets of 2-designs of size 3 is more accessible. In order to resolve this problem, one should establish the existence of large sets $\mathrm{LS}[3](2, 3^n + j, 3^{n+1} + 2)$ for $j = 0, 1, 2$ and for any $n > 3$. Concerning N-legged trades, the main question is to determine the minimum volume and minimum foundation size of a $\mathrm{T}[N](t, k, v)$.

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